CANTOR FAMILIES OF PERIODIC SOLUTIONS FOR COMPLETELY RESONANT NONLINEAR WAVE EQUATIONS

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Abstract

We prove the existence of small amplitude, $(2\pi/\omega)$-periodic in time solutions of completely resonant nonlinear wave equations with Dirichlet boundary conditions for any frequency ω belonging to a Cantor-like set of asymptotically full measure and for a new set of nonlinearities. The proof relies on a suitable Lyapunov-Schmidt decomposition and a variant of the Nash-Moser implicit function theorem. In spite of the complete resonance of the equation, we show that we can still reduce the problem to a finite-*dimensional bifurcation equation. Moreover, a new simple approach for the inversion of the linearized operators required by the Nash-Moser scheme is developed. It allows us to deal also with nonlinearities that are not odd and with finite spatial regularity.*

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1. Introduction

We consider the *completely resonant* nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} + f(x, u) = 0, \\ u(t, 0) = u(t, \pi) = 0, \end{cases} \tag{1}$$

where the nonlinearity

$$f(x, u) = a_p(x)u^p + O(u^{p+1}), \quad p \geq 2,$$

is analytic in u but is only H^1 with respect to x.

We look for small amplitude, $(2\pi/\omega)$-periodic in time solutions of equation (1) for all frequencies ω in some Cantor set of *positive measure*, actually of full density at $\omega = 1$.

Equation (1) is an infinite-dimensional Hamiltonian system possessing an elliptic equilibrium at $u = 0$. The frequencies of the linear oscillations at zero are $\omega_j = j$, $\forall j = 1, 2, \ldots$, and therefore satisfy *infinitely many resonance* relations. Any solution $v = \sum_{j\geq 1} a_j \cos(jt + \theta_j) \sin(jx)$ of the linearized equation at $u = 0$,

$$\begin{cases} u_{tt} - u_{xx} = 0, \\ u(t, 0) = u(t, \pi) = 0, \end{cases} \tag{2}$$

is 2π-periodic in time. For this reason, equation (1) is called a completely resonant Hamiltonian partial differential equation (PDE).

Existence of periodic solutions close to a completely resonant elliptic equilibrium for finite-dimensional Hamiltonian systems has been proved in the celebrated theorems of Weinstein [27], Moser [21], and Fadell and Rabinowitz [13]. The proofs are based on the classical Lyapunov-Schmidt decomposition that splits the problem into two equations: the *range equation*, solved through the standard implicit function theorem, and the *bifurcation equation*, solved via variational arguments.

For proving the existence of small amplitude periodic solutions of completely resonant Hamiltonian PDEs like (1), two main difficulties must be overcome:

(i) a “small denominators” problem that arises when solving the range equation;

(ii) the presence of an *infinite-dimensional* bifurcation equation: Which solutions v of the linearized equation (2) can be continued to solutions of the nonlinear equation (1)?

The "small denominators" problem (i) is easily explained: the eigenvalues of the operator $\partial_{tt} - \partial_{xx}$ in the spaces of functions $u(t, x)$, $(2\pi/\omega)$-periodic in time and such that, say, $u(t, \cdot) \in H_0^1(0, \pi)$ for all t, are $-\omega^2 l^2 + j^2$, $l \in \mathbf{Z}$, $j \geq 1$. Therefore, for almost every $\omega \in \mathbf{R}$, the eigenvalues accumulate to zero. As a consequence, for most ω, the inverse operator of $\partial_{tt} - \partial_{xx}$ is unbounded, and the standard implicit function theorem is not applicable.

The appearance of "small denominators" is a common feature of Hamiltonian PDEs. This problem was first solved by Kuksin [17] and Wayne [26] using Kolmogorov-Arnold-Moser (KAM) theory (other existence results of quasi-periodic solutions with KAM theory were obtained, e.g., in [9], [19], [23]; see also [18] and references therein).

In [11] Craig and Wayne introduced for Hamiltonian PDEs the Lyapunov-Schmidt reduction method and solved the range equation via a Nash-Moser implicit function technique. The major difficulty concerns the inversion of the *linearized operators* obtained at any step of the Nash-Moser iteration because the eigenvalues may be arbitrarily small. (This is the "small denominators" problem (i).) The Craig-Wayne method to control such inverses is based on the Frölich-Spencer technique in [14] and (in the wave equation with Dirichlet boundary conditions) works for nonlinearities $f(x, u)$ which can be extended to analytic, odd, periodic functions so that the Dirichlet problem on $[0, \pi]$ is equivalent to the 2π-periodic problem within the space of all odd functions. A key property exploited in this case is that the *off-diagonal* terms of the linearized operator (seen as an infinite-dimensional matrix in Fourier basis) decay exponentially fast away from the diagonal. At the end of the Nash-Moser iteration, due to the "small denominators" problem (i), the range equation is solved only for a Cantor set of parameters.

We mention that the Craig-Wayne approach has been extended by Su [25] to some case where the nonlinearity has only low Sobolev regularity (for periodic conditions) and by Bourgain [6], [7] to find also quasi-periodic solutions.

The previous results apply, for example, to nonresonant or partially resonant Hamiltonian PDEs like $u_{tt} - u_{xx} + a_1(x)u = f(x, u)$, where the bifurcation equation is finite-dimensional (2-dimensional in [11] and $2m$-dimensional in [12]). With a nondegeneracy assumption ("twist condition") the bifurcation equation is solved in [11], [12], by the implicit function theorem finding a smooth path of solutions which intersects *transversally*, for a positive measure set of frequencies, the Cantor set where also the range equation has been solved.

On the other hand, for completely resonant PDEs like (1), where $a_1(x) \equiv 0$, both small divisor difficulties and infinite-dimensional bifurcation phenomena occur. It was quoted in [10] as an important problem.

The first existence results for small amplitude periodic solutions of (1) have been obtained in* [20] for the nonlinearity $f(x,u) = u^3$ and in [3] for $f(x,u) = u^3 + O(u^5)$, imposing on the frequency ω the *strongly nonresonance* condition $|\omega l - j| \geq \gamma/l$, $\forall l \neq j$. For $0 < \gamma < 1/6$, the frequencies ω satisfying such a condition accumulate to $\omega = 1$ but form a set $\mathscr{W}_\gamma$ of zero measure. For such ω's the spectrum of $\partial_{tt} - \partial_{xx}$ does not accumulate to zero, and so the "small denominators" problem (i) is bypassed. Next, problem (ii) is solved by means of the implicit function theorem, observing that the zeroth-order bifurcation equation (which is an approximation of the exact bifurcation equation) possesses, for $f(x,u) = u^3$, nondegenerate periodic solutions (see [22]).

In [4], [5], for the same set $\mathscr{W}_\gamma$ of strongly nonresonant frequencies, existence and multiplicity of periodic solutions have been proved for any nonlinearity $f(u)$. The novelty of [4], [5] was to solve the bifurcation equation via a variational principle at fixed frequency which, jointly with min-max arguments, enables us to find solutions of (1) as critical points of the Lagrangian action functional. More precisely, the bifurcation equation is, for any fixed $\omega \in \mathscr{W}_\gamma$, the Euler-Lagrange equation of a *reduced Lagrangian action functional* which possesses nontrivial critical points of mountain pass type (see [1]; see also Remark 1.4).

Unlike [3], [4], and [5], a new feature of the results of this article is that the set of frequencies ω for which we prove existence of $(2\pi/\omega)$-periodic in time solutions of (1) has positive measure, actually has full density at $\omega = 1$.

The existence of periodic solutions for a set of frequencies of positive measure has been proved in [8] in the case of periodic boundary conditions in x and for the nonlinearity $f(x,u) = u^3 + \sum_{4 \leq j \leq d} a_j(x) u^j$, where the $a_j(x)$ are trigonometric cosine polynomials in x. The nonlinear equation $u_{tt} - u_{xx} + u^3 = 0$ possesses a continuum of small amplitude, analytic, and nondegenerate periodic solutions in the form of traveling waves $u(t,x) = \delta p_0(\omega t + x)$, where $\omega^2 = 1 + \delta^2$ and p_0 is a nontrivial 2π-periodic solution of the ordinary differential equation $p_0'' = -p_0^3$. With these properties at hand, the "small denominators" problem (i) is solved via a Nash-Moser implicit function theorem adapting the estimates of Craig and Wayne [11] for nonresonant PDEs.

Recently, the existence of periodic solutions of (1) for frequencies ω in a set of positive measure has been proved in [15] using the Lindstedt series method to solve the "small denominators" problem. The article [15] applies to odd analytic nonlinearities like $f(u) = au^3 + O(u^5)$ with $a \neq 0$ (the term u^3 guarantees a nondegeneracy

*Actually, [20] deals with the case of periodic boundary conditions in x (i.e., $u(t, x + 2\pi) = u(t,x)$).

property). The reason that $f(u)$ is odd is because the solutions are obtained as analytic sine-series in x (see Remark 1.1).

We also quote the recent article [16] on the standing wave problem for a perfect fluid under gravity and with infinite depth which leads to a nonlinear and completely resonant second-order equation.

In this article we prove the existence of $(2\pi/\omega)$-periodic solutions of the completely resonant wave equation (1) with Dirichlet boundary conditions for a set of frequencies ω with full density at $\omega = 1$ and for a new set of nonlinearities $f(x, u)$, including, for example, $f(x, u) = u^2$.

We do *not* require that $f(x, u)$ can be extended on $(-\pi, \pi) \times \mathbf{R}$ to a function $g(x, u)$, smooth with respect to u, satisfying the oddness assumption $g(-x, -u) = -g(x, u)$, and we assume only H^1-regularity in the spatial variable x (see assumption **(H)**).

To deal with these cases we develop a new approach for the inversion of the *linearized operators* which is different from the one of Craig and Wayne [11] and Bourgain [6], [7]. Our method (presented in Section 4) is quite elementary, especially requiring that the frequencies ω satisfy the Diophantine first-order Melnikov nonresonance condition of Definition 3.3 with $1 < \tau < 2$ (see comments regarding the (P)-equation in Section 1.2.2).

To handle the presence of an infinite-dimensional bifurcation equation (and the connected problems that arise in a direct application of the Craig-Wayne method; see Section 1.2.2), we perform a further finite-dimensional Lyapunov-Schmidt reduction. Under the condition that the zeroth-order bifurcation equation possesses a nondegenerate solution, we find periodic solutions of (1) for asymptotically full measure sets of frequencies.

We postpone to Section 1.2 a detailed description of our method of proof.

1.1. Main result

Normalizing the period to 2π, we look for solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x, u) = 0, \\ u(t, 0) = u(t, \pi) = 0, \end{cases} \tag{3}$$

in the Hilbert space

$$X_{\sigma,s} := \Big\{ u(t, x) = \sum_{l \in \mathbf{Z}} \exp(\mathrm{i}lt)\, u_l(x) \,\Big|\, u_l \in H_0^1\big((0, \pi), \mathbf{R}\big),\ u_l(x) = u_{-l}(x),\ \forall l \in \mathbf{Z},$$
$$\text{and } \|u\|_{\sigma,s}^2 := \sum_{l \in \mathbf{Z}} \exp(2\sigma|l|)(l^{2s} + 1)\|u_l\|_{H^1}^2 < +\infty \Big\}.$$

For $\sigma > 0$, $s \geq 0$, the space $X_{\sigma,s}$ is the space of all even, 2π-periodic in time functions with values in $H_0^1((0, \pi), \mathbf{R})$ which have a bounded analytic extension in the complex strip $|\mathrm{Im}\, t| < \sigma$ with trace function on $|\mathrm{Im}\, t| = \sigma$ belonging to $H^s\big(\mathbf{T}, H_0^1((0,\pi),\mathbf{C})\big)$.

For $2s > 1$, $X_{\sigma,s}$ is a Banach algebra with respect to multiplication of functions, namely,*

$$u_1, u_2 \in X_{\sigma,s} \implies u_1u_2 \in X_{\sigma,s} \qquad \text{and} \qquad \|u_1u_2\|_{\sigma,s} \le C\|u_1\|_{\sigma,s}\|u_2\|_{\sigma,s}.$$

It is natural to look for solutions of (3) which are even in time because equation (1) is reversible.

A weak solution $u \in X_{\sigma,s}$ of (3) is a classical solution because the map $x \mapsto u_{xx}(t,x) = \omega^2 u_{tt}(t,x) - f(x,u(t,x))$ belongs to $H_0^1(0,\pi)$ for all $t \in \mathbf{T}$, and hence, $u(t,\cdot) \in H^3(0,\pi) \subset C^2([0,\pi])$.

Remark 1.1
Let us explain why we have chosen $H_0^1((0,\pi),\mathbf{R})$ as configuration space instead of $Y := \big\{u(x) = \sum_{j\ge1} u_j \sin(jx) \,\big|\, \sum_j \exp(2aj) j^{2\rho}|u_j|^2 < +\infty\big\}$ as in [11], [15], which is natural if the nonlinearity $f(x,u)$ can be extended to an analytic in both variables odd function. For nonodd nonlinearities f (even analytic), it is not possible, in general, to find a nontrivial, smooth solution of (1) with $u(t,\cdot) \in Y$ for all t. For example, assume that $f(x,u) = u^2$. Deriving twice the equation with respect to x and using the fact that $u(t,0) = 0$, $u_{xx}(t,0) = 0$, $u_{ttxx}(t,0) = 0$, we deduce $-u_{xxxx}(t,0) + 2u_x^2(t,0) = 0$. Now $u_{xxxx}(t,0) = 0$, $\forall t$, because all the even derivatives of any function in Y vanish at $x = 0$. Hence $u_x^2(t,0) = 0$, $\forall t$. But this implies, using again the equation, that $\partial_x^k u(t,0) = 0$, $\forall k$, $\forall t$. Hence, by the analyticity of $u(t,\cdot) \in Y$, $u \equiv 0$.

The space of the solutions of the linear equation $v_{tt} - v_{xx} = 0$ which belong to $H_0^1(\mathbf{T}\times(0,\pi),\mathbf{R})$ and are even in time is

$$V := \Big\{v(t,x) = \sum_{l\ge1} 2\cos(lt)u_l\sin(lx) \,\Big|\, u_l \in \mathbf{R},\ \sum_{l\ge1} l^2|u_l|^2 < +\infty\Big\}.$$

V can also be written as

$$V := \Big\{v(t,x) = \eta(t+x) - \eta(t-x) \,\Big|\, \eta \in H^1(\mathbf{T},\mathbf{R}) \text{ with } \eta \text{ odd}\Big\}.$$

We assume that the nonlinearity f satisfies

(H) $f(x,u) = \sum_{k\ge p} a_k(x)u^k$, $p \ge 2$, and $a_k(x) \in H^1((0,\pi),\mathbf{R})$ verify $\sum_{k\ge p} \|a_k\|_{H^1}\rho^k < +\infty$ for some $\rho > 0$.

*The proof is as in [24], recalling that $H_0^1((0,\pi),\mathbf{R})$ is a Banach algebra with respect to multiplication of functions.

THEOREM 1.1
Assume that $f(x,u)$ *satisfies assumption* **(H)** *and*

$$f(x,u) = \begin{cases} a_2 u^2 + \sum_{k\geq 4} a_k(x) u^k, & a_2 \neq 0, \\ \text{or} \\ a_3(x) u^3 + \sum_{k\geq 4} a_k(x) u^k, & \langle a_3 \rangle := \left(\frac{1}{\pi}\right) \int_0^\pi a_3(x)\, dx \neq 0. \end{cases}$$

Then $s > 1/2$ *being given, there exist* $\delta_0 > 0$, $\overline{\sigma} > 0$ *and a* C^∞*-curve* $[0, \delta_0) \ni \delta \to u(\delta) \in X_{\overline{\sigma}/2,s}$ *with the following properties:*

(i) $\|u(\delta) - \delta \overline{v}\|_{\overline{\sigma}/2,s} = O(\delta^2)$ *for some* $\overline{v} \in V \cap X_{\overline{\sigma},s}$, $\overline{v} \neq \{0\}$*;*

(ii) *there exists a Cantor set* $\mathscr{C} \subset [0, \delta_0)$ *of asymptotically full measure, that is, satisfying*

$$\lim_{\eta \to 0^+} \frac{\text{meas}(\mathscr{C} \cap (0,\eta))}{\eta} = 1, \tag{4}$$

such that, $\forall\, \delta \in \mathscr{C}$, $u(\delta)$ *is a* 2π*-periodic, even in time, classical solution of (3) with, respectively,*

$$\omega = \omega(\delta) = \begin{cases} \sqrt{1 - 2\delta^2} \\ \text{or} \\ \sqrt{1 + 2\delta^2 \, \text{sign}\langle a_3 \rangle}. \end{cases}$$

As a consequence, $\forall \delta \in \mathscr{C}$, $\widetilde{u}(\delta)(t,x) := u(\delta)(\omega(\delta)t, x)$ *is a* $(2\pi/\omega(\delta))$*-periodic, even in time, classical solution of equation (1).*

By (4) also, the Cantor-like set $\{\omega(\delta) \mid \delta \in \mathscr{C}\}$ has asymptotically full measure at $\omega = 1$.

Remark 1.2
The same conclusions of Theorem 1.1 hold true also for $f(x,u) = a_4 u^4 + O(u^8)$ with $\omega^2 = 1 - 2\delta^6$. This was recently proved in [2] as a further application of the techniques of the present article (see Remark 1.5).

Theorem 1.1 is related to Theorem 1.2 stated in Section 1.2.1.

Remark 1.3
Under the hypotheses of Theorem 1.1 we could also get multiplicity of periodic solutions as a consequence of Theorem 1.2 and Lemmas 6.1 and 6.3. More precisely, there exist $n_0 \in \mathbf{N}$ and a Cantor-like set $\mathscr{C}$ of asymptotically full measure such that $\forall \delta \in \mathscr{C}$, equation (1) has a $\big(2\pi/(n\omega(\delta))\big)$-periodic solution u_n for any $n_0 \leq n \leq N(\delta)$ with $\lim_{\delta \to 0} N(\delta) = \infty$ (u_n is, in particular, $(2\pi/\omega(\delta))$-periodic). This can be seen as an analogue for (1) of the well-known multiplicity results of Weinstein [27], Moser

[21], and Fadell and Rabinowitz [13], which hold in finite dimension. Multiplicity of solutions of (1) was also obtained in [5] but only for the zero measure set of *strongly nonresonant* frequencies $\mathcal{W}_\gamma$.

1.2. The Lyapunov-Schmidt reduction

Instead of looking for solutions of (3) in a shrinking neighborhood of zero, it is a convenient device to perform the rescaling

$$u \to \delta u, \quad \delta > 0,$$

obtaining

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta^{p-1} g_\delta(x,u) = 0, \\ u(t,0) = u(t,\pi) = 0, \end{cases} \tag{5}$$

where

$$g_\delta(x,u) := \frac{f(x,\delta u)}{\delta^p} = a_p(x)u^p + \delta a_{p+1}(x)u^{p+1} + \cdots.$$

To find solutions of (5), we try to implement the Lyapunov-Schmidt reduction according to the orthogonal decomposition

$$X_{\sigma,s} = (V \cap X_{\sigma,s}) \oplus (W \cap X_{\sigma,s}),$$

where

$$W := \Big\{ w = \sum_{l \in \mathbf{Z}} \exp(\mathrm{i}lt)\, w_l(x) \in X_{0,s} \,\Big|\, w_{-l} = w_l \text{ and } \int_0^\pi w_l(x)\sin(lx)\,dx = 0,\ \forall l \in \mathbf{Z} \Big\}. \tag{6}$$

(The lth time-Fourier coefficient $w_l(x)$ must be orthogonal to $\sin(lx)$.)

Looking for solutions $u = v + w$ with $v \in V$, $w \in W$, we are led to solve the bifurcation equation (called the (Q)-equation) and the range equation (called the (P)-equation)

$$\begin{cases} -\dfrac{(\omega^2 - 1)}{2}\Delta v = \delta^{p-1}\Pi_V g_\delta(x, v+w), & (Q) \\ L_\omega w = \delta^{p-1}\Pi_W g_\delta(x, v+w), & (P) \end{cases} \tag{7}$$

where

$$\Delta v := v_{xx} + v_{tt}, \qquad L_\omega := -\omega^2 \partial_{tt} + \partial_{xx},$$

and $\Pi_V : X_{\sigma,s} \to V$, $\Pi_W : X_{\sigma,s} \to W$ denote the projectors, respectively, on V and W.

1.2.1. The zeroth-order bifurcation equation

In order to find nontrivial solutions of (7), we impose a suitable relation between the frequency ω and the amplitude δ. (As $\delta \to 0$, ω must tend to 1.)

The simplest situation occurs when

$$\Pi_V\big(a_p(x)v^p\big) \not\equiv 0. \tag{8}$$

Assumption (8) amounts to require that

$$\exists v \in V \quad \text{such that} \int_\Omega a_p(x)v^{p+1}(t,x)\,dt\,dx \neq 0, \qquad \Omega := \mathbf{T} \times (0,\pi), \tag{9}$$

which is verified if and only if $a_p(\pi - x) \not\equiv (-1)^p a_p(x)$ (see Lemma A.1).

When condition (8) (equivalently, (9)) holds, we set the *frequency-amplitude* relation

$$\frac{\omega^2 - 1}{2} = \varepsilon, \qquad |\varepsilon| := \delta^{p-1},$$

so that system (7) becomes

$$\begin{cases} -\Delta v = \Pi_V g(\delta, x, v + w), & (Q) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v + w), & (P) \end{cases} \tag{10}$$

where

$$g(\delta, x, u) := s^* g_\delta(x, u) = s^*\big(a_p(x)u^p + \delta a_{p+1}(x)u^{p+1} + \cdots\big)$$

and

$$s^* := \operatorname{sign}(\varepsilon).$$

When $\delta = 0$ (and hence, $\varepsilon = 0$), system (10) reduces to $w = 0$ and the *zeroth-order bifurcation equation*

$$-\Delta v = s^* \Pi_V\big(a_p(x)v^p\big) \tag{11}$$

which is the Euler-Lagrange equation of the functional $\Phi_0 : V \to \mathbf{R}$

$$\Phi_0(v) = \frac{\|v\|_{H^1}^2}{2} - s^* \int_\Omega a_p(x)\frac{v^{p+1}}{p+1}\,dx\,dt, \tag{12}$$

where $\|v\|_{H^1}^2 := \int_\Omega v_t^2 + v_x^2\,dx\,dt$.

By the mountain pass theorem in [1], taking

$$s^* := \begin{cases} 1, & \text{that is, } \varepsilon > 0,\ \omega > 1,\ \text{if } \exists\, v \in V \text{ such that } \int_\Omega a_p(x) v^{p+1} > 0, \\ -1, & \text{that is, } \varepsilon < 0,\ \omega < 1,\ \text{if } \exists\, v \in V \text{ such that } \int_\Omega a_p(x) v^{p+1} < 0, \end{cases} \tag{13}$$

there exists at least one nontrivial critical point of Φ_0 (i.e., a solution of (11)).

We say that a solution $\overline{v} \in V$ of equation (11) is nondegenerate if zero is the only solution of the linearized equation at $\overline{v}$ (i.e., $\ker \Phi_0''(\overline{v}) = \{0\}$).

If condition (8) is violated (as for $f(x,u) = a_2 u^2$), the right-hand side of equation (11) vanishes. In this case the correct zeroth-order nontrivial bifurcation equation involves higher-order nonlinear terms, and another *frequency-amplitude* relation is required (see Section 1.2.3).

For the sake of clarity, we develop all the details when the zeroth-order bifurcation equation is (11). In Section 6.2 we describe the changes for dealing with other cases.

We can also look for $(2\pi/n)$-time-periodic solutions of the zeroth-order bifurcation equation (11). (They are particular 2π-periodic solutions.) Let

$$\begin{aligned} V_n &:= \big\{ v \in V \;\big|\; v \text{ is } (2\pi/n)\text{-periodic in time} \big\} \\ &= \big\{ v(t,x) = \eta(nt + nx) - \eta(nt - nx) \;\big|\; \eta \in H^1(\mathbf{T}, \mathbf{R}) \text{ with } \eta \text{ odd} \big\}. \end{aligned} \tag{14}$$

If $v \in V_n$, then $\Pi_V(a_p(x) v^p) \in V_n$, and the critical points of $\Phi_{0|V_n}$ are the solutions of equation (11) which are $(2\pi/n)$-periodic. Also, $\Phi_{0|V_n}$ possesses a mountain pass critical point for any n (see [5]).

We say that a solution $\overline{v} \in V_n$ of (11) is nondegenerate in V_n if zero is the only solution in V_n of the linearized equation at $\overline{v}$ (i.e., $\ker \Phi''_{0|V_n}(\overline{v}) = \{0\}$).

THEOREM 1.2
Let f satisfy (8) and **(H)**. *Assume that $\overline{v} \in V_n$ is a nontrivial solution of the zeroth-order bifurcation equation (11) which is nondegenerate in V_n.*

Then the conclusions of Theorem 1.1 hold with $\omega = \omega(\delta) = \sqrt{1 + 2 s^* \delta^{p-1}}$.

1.2.2. About the proof of Theorem 1.2
Sections 2–5 are devoted to the proof of Theorem 1.2. Without genuine loss of generality, the proof is carried out for $n = 1$, and we explain why it works for $n > 1$ as well at the end of Section 5.

The natural way to deal with (10) is to solve first the (P)-equation (e.g., through a Nash-Moser procedure) and then to insert the solution $w(\delta, v)$ in the (Q)-equation. However, since V is infinite-dimensional here, a serious difficulty arises: if $v \in V \cap X_{\sigma_0,s}$, then the solution $w(\delta, v)$ of the range equation, obtained with any Nash-Moser iteration scheme, will have a lower regularity (e.g., $w(\delta, v) \in X_{\sigma_0/2,s}$). Therefore in solving next the bifurcation equation for $v \in V$, the best estimate we can obtain is $v \in V \cap X_{\sigma_0/2,s+2}$, which makes the scheme incoherent. Moreover, we have to ensure that the zeroth-order bifurcation equation (11) has solutions $v \in V$ which are analytic, a necessary property to initiate an analytic Nash-Moser scheme. (In [11], [12], these problems do not arise since the bifurcation equation is finite-dimensional.)

We overcome these difficulties thanks to a reduction to a *finite-dimensional* bifurcation equation on a subspace of V of dimension N independent of ω. This reduction can be implemented, in spite of the complete resonance of equation (1), thanks to the compactness of the operator $(-\Delta)^{-1}$.

We introduce the decomposition $V = V_1 \oplus V_2$, where

$$\begin{cases} V_1 := \Big\{ v \in V \;\Big|\; v(t,x) = \sum_{l=1}^{N} 2\cos(lt) u_l \sin(lx),\ \ u_l \in \mathbf{R} \Big\}, \\ V_2 := \Big\{ v \in V \;\Big|\; v(t,x) = \sum_{l \geq N+1} 2\cos(lt) u_l \sin(lx),\ \ u_l \in \mathbf{R} \Big\}. \end{cases}$$

Setting $v := v_1 + v_2$ with $v_1 \in V_1$, $v_2 \in V_2$, system (10) is equivalent to

$$\begin{cases} -\Delta v_1 = \Pi_{V_1} g(\delta, x, v_1 + v_2 + w), & (Q1) \\ -\Delta v_2 = \Pi_{V_2} g(\delta, x, v_1 + v_2 + w), & (Q2) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v_1 + v_2 + w), & (P) \end{cases} \tag{15}$$

where $\Pi_{V_i} : X_{\sigma,s} \to V_i$ $(i = 1, 2)$, denote the orthogonal projectors on V_i $(i = 1, 2)$.

Our strategy to find solutions of system (15) (and hence, to prove Theorem 1.2) is the following.

Solution of the $(Q2)$*-equation.* We solve first the $(Q2)$-equation, obtaining $v_2 = v_2(\delta, v_1, w) \in V_2 \cap X_{\sigma,s+2}$ when $w \in W \cap X_{\sigma,s}$, by the contraction mapping theorem, provided that we have chosen N large enough and $0 < \sigma \leq \overline{\sigma}$ small enough, depending on the nonlinearity f but *independent of* δ (see Section 2).

Solution of the (P)*-equation.* Next, we solve the (P)-equation, obtaining $w = w(\delta, v_1) \in W \cap X_{\overline{\sigma}/2,s}$ by means of a Nash-Moser type implicit function theorem for (δ, v_1) belonging to some Cantor-like set B_∞ of parameters (see Theorem 3.1).

Our approach for the inversion of the *linearized operators* at any step of the Nash-Moser iteration is different from the Craig-Wayne-Bourgain method. We develop $u(t,\cdot) \in H_0^1((0,\pi), \mathbf{R})$ in time-Fourier expansion only, and we distinguish the *diagonal part* $D = \operatorname{diag}\{D_k\}_{k \in \mathbf{Z}}$ of the operator that we want to invert. Next, using Sturm-Liouville theory (see Lemma 4.1), we diagonalize each D_k in a suitable basis of $H_0^1((0,\pi), \mathbf{R})$ (close to, but different from $(\sin jx)_{j \geq 1}$). Assuming a *first-order Melnikov nonresonance condition* (see Definition 3.3), we prove that its eigenvalues are polynomially bounded away from zero, and so we invert D with sufficiently good estimates (see Corollary 4.2). The presence of the *off-diagonal* Toepliz operators requires us to analyze the "small divisors": for our method, it is sufficient to prove that the product of two "small divisors" is larger than a constant if the corresponding *singular sites* are close enough (see Lemma 4.5). This holds true if the Diophantine exponent $\tau \in (1, 2)$ by the lower bound of Lemma 4.3. Moreover, for $\tau \in (1, 2)$, the nonresonance Diophantine conditions are particularly simple (see Definition 3.3 and the Cantor set B_∞ in Theorem 3.1). This restriction for the values of the exponent τ simplifies also the proof of Lemma 4.9, where the loss of derivatives due to the "small divisors" is compensated by the regularizing property of the map v_2.

Solution of the $(Q1)$-equation. Finally, in Section 5 we consider the finite-dimensional $(Q1)$-equation.

We could define a smooth functional $\Psi : [0, \delta_0) \times V_1 \to \mathbf{R}$ such that any critical point $v_1 \in V_1$ of $\Psi(\delta, \cdot)$ with $(\delta, v_1) \in B_\infty$ ($\equiv$ the Cantor-like set of parameters for which the (P)-equation is solved exactly) gives rise to an exact solution of (3) (see [4]). Moreover, it would be possible to prove the existence of a critical point $v_1(\delta)$ of $\Psi(\delta, \cdot)$, $\forall \delta > 0$ small enough, using the mountain pass theorem in [1].

However, since the section $E_\delta := \{v_1 \mid (\delta, v_1) \in B_\infty\}$ has *gaps* (except for δ in a zero measure set; see Remark 1.4), the difficulty is to prove that $(\delta, v_1(\delta)) \in B_\infty$ for a large set of δ's. Although B_∞ is in some sense a *large* set, this property is not obvious. In this article we prove that it holds at least if the path $(\delta \mapsto v_1(\delta))$ is C^1 (see Proposition 3.2) and so intersects *transversally* the Cantor set B_∞.

This is why we require in Theorem 1.2 nondegenerate solutions of the zeroth-order bifurcation equation (11). This condition enables us to use the implicit function theorem, yielding a smooth path $(\delta \to v_1(\delta))$ of solutions of the $(Q1)$-equation.

Remark 1.4
The section E_δ has *no gaps* if and only if the frequency $\omega(\delta) = \sqrt{1 + 2s^*\delta^{p-1}}$ belongs to the uncountable zero measure set $\mathcal{W}_\gamma := \{|\omega l - j| \geq \gamma/l,\ \forall j \neq l, l \geq 0, j \geq 1\}$ of [3]. This explains why in [4], [5] we had been able to prove the existence of periodic solutions for *any* nonlinearity f, solving the bifurcation equation with variational methods.

We lay the stress on the fact that the parts on the $(Q2)$- and (P)-equations do not use the nondegeneracy condition. We hope that we will be able to improve our results relaxing the nondegeneracy condition in a subsequent work, using the variational formulation of the $(Q1)$-equation and results on properties of critical sets for parameter-depending functionals.

1.2.3. About the proof of Theorem 1.1

To deduce Theorem 1.1 when $f(x,u) = a_3(x)u^3 + O(u^4)$ and $\langle a_3\rangle \neq 0$, we just have to prove that the zeroth-order bifurcation equation*

$$- \Delta v = s^* \Pi_V \big(a_3(x)v^3\big) \tag{16}$$

possesses, at least for n large, a nondegenerate solution in V_n. Choosing $s^* \in \{-1, 1\}$ so that $s^*\langle a_3\rangle > 0$, this is proved in Lemma 6.1.

In the case $f(x,u) = a_2u^2 + O(u^4)$, condition (8) is violated because $\Pi_V v^2 \equiv 0$, and we have to use a development in δ of higher order, as in [4]. Imposing in (7) the frequency-amplitude relation

$$\frac{\omega^2 - 1}{2} = -\delta^2, \tag{17}$$

the correct zeroth-order bifurcation equation turns out to be (see Section 6.2)

$$- \Delta v + 2a_2^2 \Pi_V \big(vL^{-1}(v^2)\big) = 0, \tag{18}$$

where $L^{-1} : W \to W$ is the inverse operator of $-\partial_{tt} + \partial_{xx}$. Equation (18) is the Euler-Lagrange equation of

$$\Phi_0(v) = \frac{\|v\|_{H^1}^2}{2} + \frac{a_2^2}{2} \int_\Omega v^2 L^{-1} v^2, \tag{19}$$

which again possesses mountain pass critical points because $\int_\Omega v^2 L^{-1} v^2 < 0, \forall v \in V$ (see [4]).

The existence of a *nondegenerate* critical point of $(\Phi_0)_{|V_n}$ for n large enough is proved in Lemma 6.3. This implies, as in Theorem 1.2, the conclusions of Theorem 1.1.

Remark 1.5

Also, when $f(x,u) = a_4u^4 + O(u^8)$, condition (8) is violated because $\Pi_V v^4 \equiv 0$. Imposing the frequency-amplitude relation $\omega^2 - 1 = -2\delta^6$, the correct zeroth-order bifurcation equation turns out to be

$$- \Delta v + 4a_4^2 \Pi_V \big(v^3 L^{-1}(v^4)\big) = 0. \tag{20}$$

The existence of a solution of (20) which is *nondegenerate* in V_n for n large enough is proved in [2]. This implies the conclusions of Theorem 1.1.

*Note that $\langle a_3\rangle \neq 0$ implies condition (9) because $a_3(\pi - x) \not\equiv -a_3(x)$, and so $\Pi_V(a_3(x)v^3) \not\equiv 0$.

2. Solution of the $(Q2)$-equation

The main assumption of Theorem 1.2 says that at least one of the critical points of Φ_0 defined in (12) or of the restriction of Φ_0 to some V_n, called $\overline{v}$, is nondegenerate. For definiteness, we assume that $\overline{v}$ is nondegenerate in the whole space V.

By the regularizing property of

$$(-\Delta)^{-1} : V \cap H^k(\Omega) \to V \cap H^{k+2}(\Omega), \quad \forall k \geq 0,$$

and a direct bootstrap argument, $\overline{v} \in H^k(\Omega)$, $\forall k \geq 0$. Therefore* $\overline{v} \in V \cap C^\infty(\Omega)$.

In the sequel of this article, $s > 1/2$ is fixed once and for all. We also fix some $R > 0$ such that

$$\|\overline{v}\|_{0,s} < R. \tag{21}$$

By the analyticity assumption **(H)** on the nonlinearity f and the Banach algebra property of $X_{\sigma,s}$, there is a constant $K_0 > 0$ such that

$$\begin{aligned} \|g(\delta, x, u)\|_{\sigma,s} &= \Big\| \sum_{k \geq p} a_k(x) \delta^{k-p} u^k \Big\|_{\sigma,s} \leq \sum_{k \geq p} \|a_k\|_{H^1} \delta^{k-p} K_0^{k-1} \|u\|_{\sigma,s}^k \\ &\leq C \|u\|_{\sigma,s}^p \sum_{k \geq p} \|a_k\|_{H^1} (\delta K_0 \|u\|_{\sigma,s})^{k-p} \leq C' \|u\|_{\sigma,s}^p \end{aligned} \tag{22}$$

in the open domain $\mathcal{U}_\delta := \{u \in X_{\sigma,s} \mid \delta K_0 \; \|u\|_{\sigma,s} < \rho\}$ because the power series $\sum_{k \geq p} \|a_k\|_{H^1} \rho^{k-p} < +\infty$ by **(H)**. The Nemitsky operator

$$X_{\sigma,s} \ni u \to g(\delta, x, u) \in X_{\sigma,s}$$

is in $C^\infty(\mathcal{U}_\delta, X_{\sigma,s})$. We specify that all the norms $\| \; \|_{\sigma,s}$ are equivalent on V_1. In the sequel,

$$B(\rho, V_1) := \Big\{ v_1 \in V_1 \,\Big|\, \|v_1\|_{0,s} \leq \rho \Big\}.$$

The fact that $\overline{v} \in V \cap X_{\sigma,s}$ for some $\sigma > 0$ is a consequence of the following lemma.

LEMMA 2.1 (Solution of the $(Q2)$-equation)
There exist $N \in \mathbf{N}_+$, $\overline{\sigma} := \ln 2/N > 0$, $\delta_0 > 0$ such that

(a) $\forall 0 \leq \sigma \leq \overline{\sigma}$, $\forall \|v_1\|_{0,s} \leq 2R$, $\forall \|w\|_{\sigma,s} \leq 1$, $\forall \delta \in [0, \delta_0)$, *there exists a unique* $v_2 = v_2(\delta, v_1, w) \in V_2 \cap X_{\sigma,s}$ *with* $\|v_2(\delta, v_1, w)\|_{\sigma,s} \leq 1$ *which solves the* $(Q2)$*-equation;*

*The following is true even if $a_p(x) \in H^1((0, \pi), \mathbf{R})$ only because the projection Π_V has a regularizing effect in the variable x.

(b) $v_2(0, \Pi_{V_1}\overline{v}, 0) = \Pi_{V_2}\overline{v}$;

(c) $v_2(\delta, v_1, w) \in X_{\sigma,s+2}$, *the function** $v_2(\cdot,\cdot,\cdot) \in C^\infty([0,\delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}), V_2 \cap X_{\sigma,s+2})$, *and* $D^k v_2$ *is bounded on* $[0,\delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s})$ *for any* $k \in N$;

(d) *if, in addition,* $\|w\|_{\sigma,s'} < +\infty$ *for some* $s' \geq s$, *then (provided* δ_0 *has been chosen small enough)* $\|v_2(\delta, v_1, w)\|_{\sigma,s'+2} \leq K(s', \|w\|_{\sigma,s'})$.

Proof

Fixed points of the nonlinear operator $\mathcal{N}(\delta, v_1, w, \cdot) : V_2 \to V_2$ defined by

$$\mathcal{N}(\delta, v_1, w, v_2) := (-\Delta)^{-1}\Pi_{V_2} g(\delta, x, v_1 + w + v_2)$$

are solutions of equation $(Q2)$. For $w \in W \cap X_{\sigma,s}$, $v_2 \in V_2 \cap X_{\sigma,s}$, we have $\mathcal{N}(\delta, v_1, w, v_2) \in V_2 \cap X_{\sigma,s+2}$ since $g(\delta, x, v_1 + w + v_2) \in X_{\sigma,s}$ and because of the *regularizing property* of the operator $(-\Delta)^{-1}\Pi_{V_2} : X_{\sigma,s} \to V_2 \cap X_{\sigma,s+2}$.

(a) Let $B := \{v_2 \in V_2 \cap X_{\sigma,s} \mid \|v_2\|_{\sigma,s} \leq 1\}$. We claim that there exist $N \in \mathbf{N}$, $\overline{\sigma} > 0$, and $\delta_0 > 0$ such that $\forall 0 \leq \sigma < \overline{\sigma}$, $\|v_1\|_{0,s} \leq 2R$, $\|w\|_{\sigma,s} \leq 1$, $\delta \in [0, \delta_0)$, the operator $v_2 \to \mathcal{N}(\delta, v_1, w, v_2)$ is a contraction in B; more precisely,

(i) $\|v_2\|_{\sigma,s} \leq 1 \Rightarrow \|\mathcal{N}(\delta, v_1, w, v_2)\|_{\sigma,s} \leq 1$;

(ii) $v_2, \widetilde{v}_2 \in B \Rightarrow \|\mathcal{N}(\delta, v_1, w, v_2) - \mathcal{N}(\delta, v_1, w, \widetilde{v}_2)\|_{\sigma,s} \leq (1/2)\|v_2 - \widetilde{v}_2\|_{\sigma,s}$.

Let us prove (i). For all $u \in X_{\sigma,s}$, $\|(-\Delta)^{-1}\Pi_{V_2} u\|_{\sigma,s} \leq (C/(N+1)^2)\|u\|_{\sigma,s}$, and so, $\forall \|w\|_{\sigma,s} \leq 1$, $\|v_1\|_{0,s} \leq 2R$, $\delta \in [0, \delta_0)$, using (22),

$$\begin{aligned}
\|\mathcal{N}(\delta, v_1, w, v_2)\|_{\sigma,s} &\leq \frac{C}{(N+1)^2}\|g(\delta, x, v_1 + v_2 + w)\|_{\sigma,s} \\
&\leq \frac{C'}{(N+1)^2}(\|v_1\|^p_{\sigma,s} + \|v_2\|^p_{\sigma,s} + \|w\|^p_{\sigma,s}) \\
&\leq \frac{C'}{(N+1)^2}\big(\exp(\sigma p N)\|v_1\|^p_{0,s} + \|v_2\|^p_{\sigma,s} + 1\big) \\
&\leq \frac{C'}{(N+1)^2}\big((4R)^p + \|v_2\|^p_{\sigma,s} + 1\big)
\end{aligned}$$

for $\exp(\sigma N) \leq 2$, where we have used the fact that $\|v_1\|_{\sigma,s} \leq \exp(\sigma N)\|v_1\|_{0,s} \leq 4R$.

For N large enough (depending on R), we get

$$\|v_2\|_{\sigma,s} \leq 1 \Rightarrow \|\mathcal{N}(\delta, v_1, w, v_2)\|_{\sigma,s} \leq \frac{C'}{(N+1)^2}\big((4R)^p + 1 + 1\big) \leq 1,$$

*The formula $l \in C^\infty(A, Y)$ means, if A is not open, that there is an open neighborhood U of A and an extension $\widetilde{l} \in C^\infty(U, Y)$ of l.

and (i) follows, taking $\overline{\sigma} := \ln 2/N$. Property (ii) can be proved similarly, and the existence of a unique solution $v_2(\delta, v_1, w) \in B$ follows by the contraction mapping theorem.

(b) We may assume that N has been chosen so large that $\|\Pi_{V_2}\overline{v}\|_{0,s} \leq 1/2$. Since $\overline{v}$ solves equation (11), $\Pi_{V_2}\overline{v}$ solves the $(Q2)$-equation associated with $(\delta, v_1, w) = (0, \Pi_{V_1}\overline{v}, 0)$. Since $\Pi_{V_2}\overline{v} = \mathcal{N}(0, \Pi_{V_1}\overline{v}, 0, \Pi_{V_2}\overline{v})$ and $\Pi_{V_2}\overline{v} \in B$, we deduce $\Pi_{V_2}\overline{v} = v_2(0, \Pi_{V_1}\overline{v}, 0)$.

(c) As a consequence of (ii), the linear operator $I - D_{v_2}\mathcal{N}$ is invertible at the fixed point of $\mathcal{N}(\delta, v_1, w, \cdot)$. Since the map $(\delta, v_1, w, v_2) \mapsto \mathcal{N}(\delta, v_1, w, v_2)$ is C^∞, by the implicit function theorem $v_2 : \{(\delta, v_1, w) \mid \delta \in [0, \delta_0), \|v_1\|_{0,s} \leq 2R, \|w\|_{\sigma,s} \leq 1\} \to V_2 \cap X_{\sigma,s}$ is a C^∞-map. Hence, since $(-\Delta)^{-1}\Pi_{V_2}$ is a continuous linear operator from $X_{\sigma,s}$ to $V_2 \cap X_{\sigma,s+2}$ and

$$v_2(\delta, v_1, w) = (-\Delta)^{-1}\Pi_{V_2}\big(g(\delta, x, v_1 + w + v_2(\delta, v_1, w))\big), \tag{23}$$

by the regularity of the Nemitsky operator induced by g, $v_2(\cdot, \cdot, \cdot) \in C^\infty([0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}), V_2 \cap X_{\sigma,s+2})$. The estimates for the derivatives can be obtained similarly.

(d) Let us first prove the following: if $\delta\|u\|_{\sigma,s}$ is small enough, then

$$u \in X_{\sigma,r} \Rightarrow g(\delta, x, u) \in X_{\sigma,r}, \quad \forall r \geq s. \tag{24}$$

We first observe that since $r \geq s > 1/2$, for $u, v \in H^r(\mathbf{R}/2\pi\mathbf{Z})$, we have $\|uv\|_{H^r} \leq C_r(\|u\|_\infty\|v\|_{H^r} + \|v\|_\infty\|u\|_{H^r})$. This is a consequence of the Gagliardo-Nirenberg inequalities. Hence there is a positive constant K_r such that

$$\|u^l\|_{H^r} \leq K_r^{l-1}\|u\|_\infty^{l-1}\|u\|_{H^r} \leq K_r^{l-1}\|u\|_{H^s}^{l-1}\|u\|_{H^r}, \quad \forall u \in H^r(\mathbf{R}/2\pi\mathbf{Z}), \ \forall l \geq 1.$$

Considering the extension of a function $u \in X_{\sigma,r}$ to the complex strip of width σ and using the fact that $H_0^1(0, \pi)$ is a Banach algebra, we can derive that $\forall r \geq s$, $\|u^l\|_{\sigma,r} \leq K_r^{l-1}\|u\|_{\sigma,s}^{l-1}\|u\|_{\sigma,r}$. Therefore

$$\begin{aligned}\|g(\delta, x, u)\|_{\sigma,r} &= \Big\|\sum_{k\geq p} a_k(x)\delta^{k-p}u^k\Big\|_{\sigma,r} \leq \|u\|_{\sigma,r}^p \sum_{k\geq p}\|a_k\|_{H^1}\|(\delta u)^{k-p}\|_{\sigma,r} \\ &\leq \|u\|_{\sigma,r}^p\Big[\|a_p\|_{H^1} + \sum_{k>p}\|a_k\|_{H^1}C^{k-p}(\delta\|u\|_{\sigma,s})^{k-p-1}(\delta\|u\|_{\sigma,r})\Big] < +\infty\end{aligned}$$

for $\delta\|u\|_{\sigma,s}$ small enough.

Now, assume that $\|w\|_{\sigma,s'} < +\infty$ for some $s' \geq s$. Since $v_2(\delta, v_1, w) \in X_{\sigma,s}$ solves equation (23), by a direct bootstrap argument using the regularizing properties of $(-\Delta)^{-1}\Pi_{V_2} : X_{\sigma,r} \to V_2 \cap X_{\sigma,r+2}$ and the fact that $\|v_1\|_{\sigma,r} < +\infty$, $\forall r \geq s$, we derive $v_2(\delta, v_1, w) \in X_{\sigma,s'+2}$ and $\|v_2(\delta, v_1, w)\|_{\sigma,s'+2} \leq K(s', \|w\|_{\sigma,s'})$. □

Remark 2.1

Lemma 2.1 implies, in particular, that the solution $\overline{v}$ of the zeroth-order bifurcation equation (11) is not only in $V \cap C^\infty(\Omega)$ but actually belongs to $V \cap X_{\overline{\sigma},s+2}$ and therefore is analytic in t and hence in x.

We stress that we consider as *fixed* the constants N and $\overline{\sigma}$ obtained in Lemma 2.1, which depend only on the nonlinearity f and on $\overline{v}$. On the contrary, we allow δ_0 to decrease in the next sections.

3. Solution of the (P)-equation

By the previous section we are reduced to solve the (P)-equation with $v_2 = v_2(\delta, v_1, w)$; namely,

$$L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w), \tag{25}$$

where

$$\Gamma(\delta, v_1, w)(t, x) := g\big(\delta, x, v_1(t, x) + w(t, x) + v_2(\delta, v_1, w)(t, x)\big). \tag{26}$$

The solution $w = w(\delta, v_1)$ of the (P)-equation (25) is obtained by means of a Nash-Moser implicit function theorem for (δ, v_1) belonging to a Cantor-like set of parameters.

We consider the orthogonal splitting $W = W^{(n)} \oplus W^{(n)\perp}$, where

$$W^{(n)} = \Big\{ w \in W \;\Big|\; w = \sum_{|l| \le L_n} \exp(\mathrm{i} l t)\, w_l(x) \Big\},$$
$$W^{(n)\perp} = \Big\{ w \in W \;\Big|\; w = \sum_{|l| > L_n} \exp(\mathrm{i} l t)\, w_l(x) \Big\}, \tag{27}$$

and L_n are integer numbers. (We choose $L_n = L_0 2^n$ with $L_0 \in \mathbf{N}$ large enough.) We denote by

$$P_n : W \to W^{(n)} \qquad \text{and} \qquad P_n^\perp : W \to W^{(n)\perp}$$

the orthogonal projectors onto $W^{(n)}$ and $W^{(n)\perp}$.

The convergence of the recursive scheme is based on properties (P1), (P2), and (P3).

(P1) (*Regularity*) $\Gamma(\cdot, \cdot, \cdot,) \in C^\infty([0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}), X_{\sigma,s})$. Moreover, $D^k\Gamma$ is bounded on $[0, \delta_0) \times B(2R, V_1) \times B(1; W \cap X_{\sigma,s})$ for any $k \in N$.

(P1) is a consequence of the C^∞-regularity of the Nemitsky operator induced by $g(\delta, x, u)$ on $X_{\sigma,s}$ and of the C^∞-regularity of the map $v_2(\cdot, \cdot, \cdot)$ proved in Lemma 2.1.

(P2) (*Smoothing estimate*) For all $w \in W^{(n)\perp} \cap X_{\sigma,s}$ and $\forall\, 0 \le \sigma' \le \sigma$, $\|w\|_{\sigma',s} \le \exp^{(-L_n(\sigma - \sigma'))} \|w\|_{\sigma,s}$.

The standard property (P2) follows from

$$\begin{aligned}\|w\|^2_{\sigma',s} &= \sum_{|l|>L_n} \exp(2\sigma'|l|)(l^{2s}+1)\|w_l\|^2_{H^1}\\ &= \sum_{|l|>L_n} \exp\big(-2(\sigma-\sigma')|l|\big)\exp(2\sigma|l|)(l^{2s}+1)\|w_l\|^2_{H^1}\\ &\le \exp\big(-2(\sigma-\sigma')L_n\big)\|w\|^2_{\sigma,s}.\end{aligned}$$

The next property (P3) is an *invertibility property* of the linearized operator $\mathscr{L}_n(\delta, v_1, w) : W^{(n)} \to W^{(n)}$ defined by

$$\mathscr{L}_n(\delta, v_1, w)[h] := L_\omega h - \varepsilon P_n \Pi_W D_w \Gamma(\delta, v_1, w)[h]. \tag{28}$$

Throughout the proof, w is the approximate solution obtained at a given step of the Nash-Moser iteration.

The invertibility of $\mathscr{L}_n(\delta, v_1, w)$ is obtained by excising the set of parameters (δ, v_1) for which zero is an eigenvalue of $\mathscr{L}_n(\delta, v_1, w)$. Moreover, in order to have bounds for the norm of the inverse operator $\mathscr{L}_n^{-1}(\delta, v_1, w)$ which are sufficiently good for the recursive scheme, we also excise the parameters (δ, v_1) for which the eigenvalues of $\mathscr{L}_n(\delta, v_1, w)$ are too small.

We prefix some definitions.

Definition 3.1 (Mean value)
For $\Omega := \mathbf{T} \times (0, \pi)$, we define

$$M(\delta, v_1, w) := \frac{1}{|\Omega|}\int_\Omega \partial_u g\big(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x)\big)\, dx\, dt.$$

Note that $M(\cdot,\cdot,\cdot) : [0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}) \to \mathbf{R}$ is a C^∞-function.

Definition 3.2
For $1 < \tau < 2$, we define

$$[w]_{\sigma,s} := \inf\Big\{ \sum_{i=0}^{q} \frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i - \sigma)^{2(\tau-1)/\beta}};\ q \ge 1,\ \overline{\sigma} \ge \sigma_i > \sigma,\ \ h_i \in W^{(i)},\ w = \sum_{i=0}^{q} h_i \Big\},$$

where $\beta := (2-\tau)/\tau$, and we set $[w]_{\sigma,s} := \infty$ if the above set is empty.

Definition 3.3 (First-order Melnikov nonresonance condition)
Let $0 < \gamma < 1$, and let $1 < \tau < 2$. We define (recall that $\omega = \sqrt{1 + 2s^*\delta^{p-1}}$ and $\varepsilon = s^*\delta^{p-1}$)

$$\Delta_n^{\gamma,\tau}(v_1, w) := \Big\{\delta \in [0, \delta_0) \;\Big|\; |\omega l - j| \geq \frac{\gamma}{(l+j)^\tau}, \Big|\omega l - j - \varepsilon \frac{M(\delta, v_1, w)}{2j}\Big| \geq \frac{\gamma}{(l+j)^\tau}, \; \forall l \in \mathbf{N}, \; j \geq 1, \; l \neq j, \; \frac{1}{3|\varepsilon|} < l, \; l \leq L_n, \; j \leq 2L_n\Big\}.$$

The set $\Delta_n^{\gamma,\tau}(v_1, w)$ contains a whole interval $[0, \eta_n)$ for some $\eta_n > 0$ small enough. (Note that $\Delta_n^{\gamma,\tau}(v_1, w)$ is defined by a finite set of inequalities.)

Remark 3.1
The intersections of the sets $\Delta_n^{\gamma,\tau}(v_1, w)$ over all possible (v_1, w) in a neighborhood of zero and over all n contains, for $|\varepsilon|\gamma^{-1}$ small, the zero measure, uncountable set $\mathscr{W}_\gamma := \{\omega \in \mathbf{R} \mid |\omega l - j| \geq \gamma/l, \forall l \neq j, l \geq 0, j \geq 1\}$, $0 < \gamma < 1/6$ introduced in [3] (see Remark 1.4 for consequences on the existence of periodic solutions).

We claim the following.

(P3) (*Invertibility of* $\mathscr{L}_n$) There exist positive constants μ, δ_0 such that if $[w]_{\sigma,s} \leq \mu$, $\|v_1\|_{0,s} \leq 2R$, and $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ for some $0 < \gamma < 1$, $1 < \tau < 2$, then $\mathscr{L}_n(\delta, v_1, w)$ is invertible and the inverse operator $\mathscr{L}_n^{-1}(\delta, v_1, w) : W^{(n)} \to W^{(n)}$ satisfies

$$\|\mathscr{L}_n^{-1}(\delta, v_1, w)[h]\|_{\sigma,s} \leq \frac{C}{\gamma}(L_n)^{\tau-1}\|h\|_{\sigma,s} \tag{29}$$

for some positive constant $C > 0$.

Property (P3) is the real core of the convergence proof and where the analysis of the "small divisors" enters into play. Property (P3) is proved in Section 4.

3.1. The Nash-Moser scheme

We are going to define recursively a sequence $\{w_n\}_{n\geq 0}$ with $w_n = w_n(\delta, v_1) \in W^{(n)}$, defined on smaller and smaller sets of *nonresonant* parameters (δ, v_1), $A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1 \subseteq A_0 := \{(\delta, v_1) \mid \delta \in [0, \delta_0), \; \|v_1\|_{0,s} \leq 2R\}$. The sequence $(w_n(\delta, v_1))$ converges to a solution $w(\delta, v_1)$ of the (P)-equation (25) for $(\delta, v_1) \in A_\infty := \bigcap_{n\geq 1} A_n$. The main goal of the construction is to show that, at the end of the recurrence, the set of parameters $A_\infty := \bigcap_{n\geq 1} A_n$ for which we have the solution $w(\delta, v_1)$ remains sufficiently large.

We define inductively the sequence $\{w_n\}_{n\geq 0}$. Define the *loss of analyticity* γ_n by

$$\gamma_n := \frac{\gamma_0}{n^2+1}, \qquad \sigma_0 = \overline{\sigma}, \qquad \sigma_{n+1} = \sigma_n - \gamma_n, \quad \forall\, n \geq 0,$$

where we choose $\gamma_0 > 0$ small such that the *total loss of analyticity*

$$\sum_{n\geq 0} \gamma_n = \sum_{n\geq 0} \frac{\gamma_0}{(n^2+1)} \leq \frac{\overline{\sigma}}{2}; \quad \text{that is, } \sigma_n \geq \frac{\overline{\sigma}}{2} > 0, \; \forall n.$$

We also assume

$$L_n := L_0 2^n, \quad \forall\, n \geq 0,$$

for some large integer L_0 specified in the next proposition.

PROPOSITION 3.1 (Induction)
Let $A_0 := \{(\delta, v_1) \mid \delta \in [0, \delta_0),\ \|v_1\|_{0,s} \leq 2R\}$*. There exists* $L_0 := L_0(\gamma, \tau) > 0$*,* $\varepsilon_0 := \varepsilon_0(\gamma, \tau) > 0$*, such that for* $\delta_0^{p-1}\gamma^{-1} < \varepsilon_0$*, there exists a sequence* $\{w_n\}_{n\geq 0}$*,* $w_n = w_n(\delta, v_1) \in W^{(n)}$*, of solutions of the equation*

$$L_\omega w_n - \varepsilon P_n \Pi_W \Gamma(\delta, v_1, w_n) = 0, \qquad (P_n)$$

defined inductively for $(\delta, v_1) \in A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1 \subseteq A_0$*, where*

$$A_n := \Big\{(\delta, v_1) \in A_{n-1} \,\Big|\, \delta \in \Delta_n^{\gamma,\tau}(v_1, w_{n-1})\Big\} \subseteq A_{n-1}, \qquad (30)$$

$w_n(\delta, v_1) = \sum_{i=0}^{n} h_i(\delta, v_1)$*, and* $h_i = h_i(\delta, v_1) \in W^{(i)}$ *satisfy* $\|h_0\|_{\sigma_0,s} \leq |\varepsilon| K_0$*,* $\|h_i\|_{\sigma_i,s} \leq |\varepsilon|\gamma^{-1} \exp(-\chi^i)$ $\forall 1 \leq i \leq n$ *for some* $1 < \chi < 2$ *and some constant* $K_0 > 0$.

We define

$$A_\infty := \bigcap_{n\geq 0} A_n.$$

Remark 3.2
For a given (δ, v_1), the sequence (w_n) may be finite because the iterative process stops after w_{k-1} if $\delta \notin \Delta_k^{\gamma,\tau}(v_1, w_{k-1})$, that is, if $(\delta, v_1) \notin A_k$. However, from this possibly finite sequence, we define a C^∞-map $\widetilde{w}(\delta, v_1)$ on the whole set A_0 (see Lemma 3.3) and Cantor-like set B_∞ such that $B_\infty \subset A_\infty$, and $\forall (\delta, v_1) \in B_\infty$, $\widetilde{w}(\delta, v_1)$ is an exact solution of the (P)-equation. It is justified in Proposition 3.2 that B_∞ is a *large* set. As a consequence also, A_∞ is *large*.

Proof of Proposition 3.1
The proof proceeds by induction.

First step: Initialization. Let L_0 be given. If $|\omega - 1| L_0 \leq 1/2$, then $L_{\omega|W^{(0)}}$ is invertible and $\|L_\omega^{-1} h\|_{\sigma_0,s} \leq 2\|h\|_{\sigma_0,s}$, $\forall h \in W^{(0)}$. Indeed, the eigenvalues of $L_{\omega|W^{(0)}}$ are $-\omega^2 l^2 + j^2$, $\forall\, 0 \leq l \leq L_0$, $j \geq 1$, $j \neq l$, and

$$|-\omega^2 l^2 + j^2| = |-\omega l + j|(\omega l + j) \geq (|j - l| - |\omega - 1| L_0)(\omega l + j) \geq \Big(1 - \frac{1}{2}\Big).$$

By the implicit function theorem, using property (P1), there exist $K_0 > 0$, $\varepsilon_1 := \varepsilon_1(\gamma, L_0) > 0$ such that if $|\varepsilon|\gamma^{-1} < \varepsilon_1$ and $\forall v_1 \in B(2R, V_1)$, equation (P_0) has a unique solution $w_0(\delta, v_1)$ satisfying

$$\|w_0(\delta, v_1)\|_{\sigma_0,s} \le K_0|\varepsilon|.$$

Moreover, for $\delta_0^{p-1}\gamma^{-1} < \varepsilon_0$, the map $(\delta, v_1) \mapsto w_0(\delta, v_1)$ is in $C^\infty(A_0, W^{(0)})$ and $\|D^k w_0(\delta, v_1)\|_{\sigma_0,s} \le C(k)$.

Second step: Iteration. Fix some $\chi \in (1, 2)$. Let $\varepsilon_2 := \varepsilon_2(L_0, \gamma, \tau) \in (0, \varepsilon_1(\gamma, L_0))$ be small enough such that

$$\varepsilon_2 \max(1, eK_0\gamma) \sum_{i\ge 0} \exp(-\chi^i)\Big(\frac{1+i^2}{\gamma_0}\Big)^{(2(\tau-1))/\beta} < \mu, \tag{31}$$

where μ is defined in property (P3) and $\beta := (2-\tau)/\tau$.

Suppose that we have already defined a solution $w_n = w_n(\delta, v_1) \in W^{(n)}$ of equation (P_n) satisfying the properties stated in the proposition. We want to define

$$w_{n+1} = w_{n+1}(\delta, v_1) := w_n(\delta, v_1) + h_{n+1}(\delta, v_1), \quad h_{n+1}(\delta, v_1) \in W^{(n+1)},$$

as an *exact* solution of the equation

$$L_\omega w_{n+1} - \varepsilon P_{n+1}\Pi_W \Gamma(\delta, v_1, w_{n+1}) = 0. \tag{P_{n+1}}$$

In order to find a solution $w_{n+1} = w_n + h_{n+1}$ of equation (P_{n+1}), we write, for $h \in W^{(n+1)}$,

$$\begin{aligned} & L_\omega(w_n + h) - \varepsilon P_{n+1}\Pi_W\Gamma(\delta, v_1, w_n + h) \\ &\quad = L_\omega w_n - \varepsilon P_{n+1}\Pi_W\Gamma(\delta, v_1, w_n) \\ &\qquad + L_\omega h - \varepsilon P_{n+1}\Pi_W D_w\Gamma(\delta, v_1, w_n)[h] + R(h) \\ &\quad = r_n + \mathscr{L}_{n+1}(\delta, v_1, w_n)[h] + R(h), \end{aligned} \tag{32}$$

where since w_n solves equation (P_n),

$$\begin{cases} r_n := L_\omega w_n - \varepsilon P_{n+1}\Pi_W\Gamma(\delta, v_1, w_n) = -\varepsilon P_n^\perp P_{n+1}\Pi_W\Gamma(\delta, v_1, w_n) \in W^{(n+1)}, \\ R(h) := -\varepsilon P_{n+1}\Pi_W\big(\Gamma(\delta, v_1, w_n + h) - \Gamma(\delta, v_1, w_n) - D_w\Gamma(\delta, v_1, w_n)[h]\big). \end{cases}$$

The term r_n is *super-exponentially* small because, using properties (P2) and (P1),

$$\begin{aligned} \|r_n\|_{\sigma_{n+1},s} &\le |\varepsilon|\; C\exp(-L_n\gamma_n)\|P_{n+1}\Pi_W\Gamma(\delta, v_1, w_n)\|_{\sigma_n,s} \\ &\le |\varepsilon|\; C'\exp(-L_n\gamma_n)\|\Gamma(\delta, v_1, w_n)\|_{\sigma_n,s} \\ &\le |\varepsilon|\; C''\exp(-L_n\gamma_n) \end{aligned} \tag{33}$$

being $\|w_n\|_{\sigma_n,s}$ bounded independently of n since, by the induction hypothesis,

$$\|w_n\|_{\sigma_n,s} \le \sum_{i=0}^{n} \|h_i\|_{\sigma_i,s} \le \max(1, eK_0\gamma)|\varepsilon|\gamma^{-1} \sum_{i=0}^{\infty} \exp(-\chi^i), \tag{34}$$

with $h_0 := w_0$. The term $R(h)$ is *quadratic* in h since, by property (P1) and (34),

$$\begin{cases} \|R(h)\|_{\sigma_{n+1},s} \le C|\varepsilon| \; \|h\|^2_{\sigma_{n+1},s}, \\ \|R(h) - R(h')\|_{\sigma_{n+1},s} \le C|\varepsilon| \; (\|h\|_{\sigma_{n+1},s} + \|h'\|_{\sigma_{n+1},s}) \; \|h - h'\|_{\sigma_{n+1},s} \end{cases} \tag{35}$$

for all $h, h' \in W^{(n+1)}$ with $\|h\|_{\sigma_{n+1},s}$, $\|h'\|_{\sigma_{n+1},s}$ small enough.

Since $w_n = \sum_{i=0}^{n} h_i$ with $\|h_i\|_{\sigma_i,s} \le \max(1, eK_0\gamma)|\varepsilon|\gamma^{-1} \; \exp(-\chi^i)$, and $\sigma_i - \sigma_{n+1} \ge \gamma_i := \gamma_0/(1+i^2)$, $\forall i = 0, \dots, n$,

$$[w_n]_{\sigma_{n+1},s} \le \sum_{i=0}^{n} \frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i - \sigma_{n+1})^{2(\tau-1)/\beta}} \le \max(1, eK_0\gamma)\frac{|\varepsilon|}{\gamma} \; \sum_{i\ge 0} \exp(-\chi^i)\Big(\frac{1+i^2}{\gamma_0}\Big)^{2(\tau-1)/\beta} < \mu$$

for $|\varepsilon|\gamma^{-1} \le \varepsilon_2$ and by (31).

Hence, by property (P3), the linear operator $\mathscr{L}_{n+1}(\delta, v_1, w_n) : D\mathscr{L}_{n+1} \subset W^{(n+1)} \to W^{(n+1)}$ is invertible for (δ, v_1) restricted to the set of parameters

$$A_{n+1} := \Big\{(\delta, v_1) \in A_n \;\Big|\; \delta \in \Delta^{\gamma,\tau}_{n+1}(v_1, w_n)\Big\} \subseteq A_n, \tag{36}$$

and the inverse operator satisfies

$$\|\mathscr{L}_{n+1}(\delta, v_1, w_n)^{-1}\|_{\sigma_{n+1},s} \le \frac{C}{\gamma}(L_{n+1})^{\tau-1}, \quad \forall(\delta, v_1) \in A_{n+1}. \tag{37}$$

By (32), equation (P_{n+1}) for $w_{n+1} = w_n + h$ is equivalent to find $h \in W^{(n+1)}$ solving

$$h = -\mathscr{L}_{n+1}(\delta, v_1, w_n)^{-1}\big(r_n + R(h)\big),$$

namely, to look for a fixed point

$$h = \mathscr{G}(\delta, v_1, w_n, h), \quad h \in W^{(n+1)}, \tag{38}$$

of the nonlinear operator

$$\mathscr{G}(\delta, v_1, w_n, \cdot) : W^{(n+1)} \to W^{(n+1)},$$
$$\mathscr{G}(\delta, v_1, w_n, h) := -\mathscr{L}_{n+1}(\delta, v_1, w_n)^{-1}\big(r_n + R(h)\big).$$

To complete the proof of the proposition, we need the following lemma.

LEMMA 3.1 (Contraction)
There exist $L_0(\gamma,\tau) > 0$, $\varepsilon_0(L_0,\gamma,\tau)$, *such that*, $\forall |\varepsilon|\gamma^{-1} < \varepsilon_0$, *the operator* $\mathcal{G}(\delta, v_1, w_n, \cdot)$ *is, for any* $n \geq 0$, *a contraction in the ball*

$$B(\rho_{n+1}; W^{(n+1)}) := \Big\{ h \in W^{(n+1)} \,\Big|\, \|h\|_{\sigma_{n+1},s} \leq \rho_{n+1} := \frac{|\varepsilon|}{\gamma} \exp(-\chi^{n+1}) \Big\}.$$

Proof
We first prove that $\mathcal{G}(\delta, v_1, w_n, \cdot)$ maps the ball $B(\rho_{n+1}; W^{(n+1)})$ into itself.

By (37), (33), and (35),

$$\begin{aligned} \|\mathcal{G}(\delta, v_1, w_n, h)\|_{\sigma_{n+1},s} &= \big\| \mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}\big(r_n + R(h)\big)\big\|_{\sigma_{n+1},s} \\ &\leq \frac{C}{\gamma}(L_{n+1})^{\tau-1}\big(\|r_n\|_{\sigma_{n+1},s} + \|R(h)\|_{\sigma_{n+1},s}\big) \\ &\leq \frac{C'}{\gamma}(L_{n+1})^{\tau-1}\big(|\varepsilon| \exp(-L_n\gamma_n) + |\varepsilon|\, \|h\|^2_{\sigma_{n+1},s}\big). \end{aligned} \tag{39}$$

By (39), if $\|h\|_{\sigma_{n+1},s} \leq \rho_{n+1}$, then

$$\|\mathcal{G}(\delta, v_1, w_n, h)\|_{\sigma_{n+1},s} \leq \frac{C'}{\gamma}(L_{n+1})^{\tau-1}|\varepsilon|\big(\exp(-L_n\gamma_n) + \rho^2_{n+1}\big) \leq \rho_{n+1},$$

provided that

$$C'\frac{|\varepsilon|}{\gamma}(L_{n+1})^{\tau-1}\exp(-L_n\gamma_n) \leq \frac{\rho_{n+1}}{2} \qquad \text{and} \qquad C'\frac{|\varepsilon|}{\gamma}(L_{n+1})^{\tau-1}\rho_{n+1} \leq \frac{1}{2}. \tag{40}$$

The first inequality in (40) becomes, for $\rho_{n+1} := |\varepsilon|\gamma^{-1} \exp(-\chi^{n+1})$,

$$C'(L_{n+1})^{\tau-1} \exp(-L_n\gamma_n) \leq \frac{1}{2} \exp(-\chi^{n+1}),$$

which, for $L_n := L_0 2^n$, $\gamma_n := \gamma_0/(1+n^2)$, and $L_0 := L_0(\gamma,\tau) > 0$ large enough, is satisfied $\forall n \geq 0$.

Next, the second inequality in (40) becomes

$$C'\frac{|\varepsilon|^2}{\gamma^2}\big(L_0(\gamma,\tau)2^{n+1}\big)^{\tau-1} \exp(-\chi^{n+1}) \leq \frac{1}{2},$$

which is satisfied for $|\varepsilon|\gamma^{-1} \leq \varepsilon_0(L_0,\gamma,\tau)$ ($\leq \varepsilon_2$) small enough, $\forall n \geq 0$.

With similar estimates, using (35), we can prove that $\forall h, h' \in B(\rho_{n+1}; W^{(n+1)})$,

$$\|\mathcal{G}(\delta, v_1, w_n, h') - \mathcal{G}(\delta, v_1, w_n, h)\|_{\sigma_{n+1},s} \leq \frac{1}{2}\|h - h'\|_{\sigma_{n+1},s}$$

again for L_0 large enough and $|\varepsilon|\gamma^{-1} \le \varepsilon_0(L_0, \gamma, \tau)$ small enough, uniformly in n, and we conclude that $\mathscr{G}(\delta, v_1, w_n, \cdot)$ is a contraction on $B(\rho_{n+1}; W^{(n+1)})$. □

By the standard contraction mapping theorem, we deduce the existence, for $L_0(\gamma, \tau)$ large enough and $|\varepsilon|\gamma^{-1} < \varepsilon_0(L_0, \gamma, \tau)$, of a unique $h_{n+1} \in W^{(n+1)}$ solving (38) and satisfying

$$\|h_{n+1}\|_{\sigma_{n+1},s} \le \rho_{n+1} = \frac{|\varepsilon|}{\gamma} \exp(-\chi^{n+1}).$$

Summarizing, $w_{n+1}(\delta, v_1) = w_n(\delta, v_1) + h_{n+1}(\delta, v_1)$ is a solution in $W^{(n+1)}$ of equation (P_{n+1}), defined for $(\delta, v_1) \in A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_1 \subseteq A_0$, and $w_{n+1}(\delta, v_1) = \sum_{i=0}^{n+1} h_i(\delta, v_1)$, where $h_i = h_i(\delta, v_1) \in W^{(i)}$ satisfy $\|h_i\|_{\sigma_i,s} \le |\varepsilon|\gamma^{-1} \exp(-\chi^i)$ for some $\chi \in (1, 2)$, $\forall i = 1, \ldots, n+1$, $\|h_0\|_{\sigma_0,s} \le K_0|\varepsilon|$. □

Remark 3.3
A difference with respect to the usual *quadratic* Nash-Moser scheme is that $h_n(\delta, v_1)$ is found as an exact solution of equation (P_n) and not just a solution of the linearized equation $r_n + \mathscr{L}_{n+1}(\delta, v_1, w_n)[h] = 0$. It appears to be more convenient to prove the regularity of $h_n(\delta, v_1)$ with respect to the parameters (δ, v_1) (see Lemma 3.2).

COROLLARY 3.1 (Solution of the (P)-equation)
For $(\delta, v_1) \in A_\infty := \bigcap_{n\ge 0} A_n$, $\sum_{i\ge 0} h_i(\delta, v_1)$ converges in $X_{\overline{\sigma}/2,s}$ to a solution $w(\delta, v_1) \in W \cap X_{\overline{\sigma}/2,s}$ of the (P)-equation (25) and $\|w(\delta, v_1)\|_{\overline{\sigma}/2,s} \le C|\varepsilon|\gamma^{-1}$. The convergence is uniform in A_∞.

Proof
By Proposition 3.1, for $(\delta, v_1) \in A_\infty := \bigcap_{n\ge 0} A_n$,

$$\sum_{i=0}^{\infty} \|h_i(\delta, v_1)\|_{\overline{\sigma}/2,s} \le \sum_{i=0}^{\infty} \|h_i(\delta, v_1)\|_{\sigma_i,s} \le \max(1, eK_0\gamma) \sum_{i=0}^{\infty} \frac{|\varepsilon|}{\gamma} \exp(-\chi^i) < +\infty. \tag{41}$$

Hence the series of functions $w = \sum_{i\ge 0} h_i : A_\infty \to W \cap X_{\overline{\sigma}/2,s}$ converges normally, and by (41), $\|w(\delta, v_1)\|_{\overline{\sigma}/2,s} \le C|\varepsilon|\gamma^{-1}$ with $C := \max(1, eK_0\gamma) \sum_{i=0}^{\infty} \exp(-\chi^i)$.

Let us justify the fact that $L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w)$. Since w_n solves equation (P_n),

$$L_\omega w_n = \varepsilon P_n \Pi_W \Gamma(\delta, v_1, w_n) = \varepsilon \Pi_W \Gamma(\delta, v_1, w_n) - \varepsilon P_n^\perp \Pi_W \Gamma(\delta, v_1, w_n). \tag{42}$$

We have, by (P2), (P1), and since $\sigma_n - (\overline{\sigma}/2) \ge \gamma_n := \gamma_0/(n^2+1)$,

$$\|P_n^\perp \Pi_W \Gamma(\delta, v_1, w_n)\|_{\overline{\sigma}/2,s} \le C \exp\left(-L_n\left(\sigma_n - \left(\frac{\overline{\sigma}}{2}\right)\right)\right) \le C \exp\left(-\gamma_0 \frac{L_0 2^n}{(n^2+1)}\right).$$

Hence, by (P1), the right-hand side in (42) converges in $X_{\overline{\sigma}/2,s}$ to $\Gamma(\delta, v_1, w)$. Moreover, since $(w_n) \to w$ in $X_{\overline{\sigma}/2,s}$, $(L_\omega w_n) \to L_\omega w$ in the sense of distributions. Hence $L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w)$. □

3.2. C^∞-extension

Before proving the key property (P3) on the linearized operator, we prove a *Whitney-differentiability* property for $w(\delta, v_1)$ extending $w(\cdot, \cdot)$ in a C^∞-way on the whole A_0.

For this, some bound on the derivatives of $h_n = w_n - w_{n-1}$ is required.

LEMMA 3.2 (Estimates for the derivatives of h_n and w_n)
For $\varepsilon_0\gamma^{-1} = \delta_0^{p-1}\gamma^{-1}$ small enough, the function $(\delta, v_1) \to h_n(\delta, v_1)$ is in $C^\infty(A_n, W^{(n)})$, $\forall n \geq 0$, and the kth-derivative $D^k h_n(\delta, v_1)$ satisfies

$$\|D^k h_n(\delta, v_1)\|_{\sigma_n,s} \leq K_1(k, \overline{\chi})^n \exp(-\overline{\chi}^n) \tag{43}$$

for $\overline{\chi} \in (1, \chi)$ and a suitable positive constant $K_1(k, \overline{\chi})$, $\forall n \geq 0$.

As a consequence, the function $(\delta, v_1) \to w_n(\delta, v_1) = \sum_{i=0}^n h_i(\delta, v_1)$ is in $C^\infty(A_n, W^{(n)})$, and the kth-derivative $D^k w_n(\delta, v_1)$ satisfies

$$\|D^k w_n(\delta, v_1)\|_{\sigma_n,s} \leq K_2(k) \tag{44}$$

for a suitable positive constant $K_2(k)$.

Proof

By the first step in the proof of Proposition 3.1, $h_0 = w_0$ depends smoothly on (δ, v_1), and $\|D^k w_0(\delta, v_1)\|_{\sigma_0,s} \leq C(k)$.

Next, assume by induction that $h_n = h_n(\delta, v_1)$ is a C^∞-map defined in A_n. We prove that $h_{n+1} = h_{n+1}(\delta, v_1)$ is C^∞ too.

First, recall that $h_{n+1} = h_{n+1}(\delta, v_1)$ is defined, in Proposition 3.1, for $(\delta, v_1) \in A_{n+1}$ as a solution in $W^{(n+1)}$ of equation (P_{n+1}); namely,

$$U_{n+1}\big(\delta, v_1, h_{n+1}(\delta, v_1)\big) = 0, \tag{P_{n+1}}$$

where

$$U_{n+1}(\delta, v_1, h) := L_\omega(w_n + h) - \varepsilon P_{n+1}\Pi_W\Gamma(\delta, v_1, w_n + h).$$

We claim that $D_h U_{n+1}(\delta, v_1, h_{n+1}) = \mathscr{L}_{n+1}(\delta, v_1, w_{n+1})$ is invertible and

$$\big\|\big(D_h U_{n+1}(\delta, v_1, h_{n+1})\big)^{-1}\big\|_{\sigma_{n+1},s} = \|\mathscr{L}_{n+1}(\delta, v_1, w_{n+1})^{-1}\|_{\sigma_{n+1},s} \leq \frac{C'}{\gamma}(L_{n+1})^{\tau-1}. \tag{45}$$

Now equation (P_{n+1}) can be written as $h + q_{n+1}(\delta, v_1, h) = 0$, where

$$q_{n+1}(\delta, v_1, h) = (\Delta)^{-1}[L_\omega w_n - (\omega^2 + 1)h_{tt} - \varepsilon P_{n+1}\Pi_W\Gamma(\delta, v_1, w_n + h)].$$

The map $q_{n+1} : [0, \delta_0) \times V_1 \times W^{(n+1)} \to W^{(n+1)}$ is C^∞, and the invertibility of $\mathscr{L}_{n+1}(\delta, v_1, w_{n+1})$ implies the injectivity and, hence (noting that $D_h q_{n+1}(\delta, v_1, h_{n+1})$ is compact), the invertibility of $I + D_h q_{n+1}(\delta, v_1, h_{n+1})$. As a consequence, by the implicit function theorem, the map $(\delta, v_1) \mapsto h_{n+1}(\delta, v_1)$ is in $C^\infty(A_{n+1}, W^{(n+1)})$.

Let us prove (45). Using (P1) and $\|w_{n+1} - w_n\|_{\sigma_{n+1},s} = \|h_{n+1}\|_{\sigma_{n+1},s} \le |\varepsilon|\gamma^{-1}\exp(-\chi^{n+1})$, we get

$$\begin{aligned}
&\|\mathscr{L}_{n+1}(\delta, v_1, w_{n+1}) - \mathscr{L}_{n+1}(\delta, v_1, w_n)\|_{\sigma_{n+1},s}\\
&\quad = \left\|\varepsilon P_{n+1}\Pi_W\big(D_w\Gamma(\delta, v_1, w_{n+1}) - D_w\Gamma(\delta, v_1, w_n)\big)\right\|_{\sigma_{n+1},s}\\
&\quad \le C|\varepsilon|\,\|h_{n+1}\|_{\sigma_{n+1},s} \le C\frac{\varepsilon^2}{\gamma}\exp(-\chi^{n+1}).
\end{aligned} \tag{46}$$

Therefore

$$\begin{aligned}
&\mathscr{L}_{n+1}(\delta, v_1, w_{n+1})\\
&\quad = \mathscr{L}_{n+1}(\delta, v_1, w_n)\Big[\mathrm{Id} + \mathscr{L}_{n+1}(\delta, v_1, w_n)^{-1}\Big(\mathscr{L}_{n+1}(\delta, v_1, w_{n+1}) - \mathscr{L}_{n+1}(\delta, v_1, w_n)\Big)\Big]
\end{aligned} \tag{47}$$

is invertible whenever (recall (37), (46))

$$\begin{aligned}
\Big\|\mathscr{L}_{n+1}(\delta, v_1, w_n)^{-1}\Big(\mathscr{L}_{n+1}(\delta, v_1, w_{n+1}) - \mathscr{L}_{n+1}(\delta, v_1, w_n)\Big)\Big\|_{\sigma_{n+1},s} &\le \frac{C}{\gamma}(L_{n+1})^{\tau-1}\frac{\varepsilon^2}{\gamma}\exp(-\chi^{n+1})\\
&< \frac{1}{2},
\end{aligned} \tag{48}$$

which is true, provided that $|\varepsilon|\gamma^{-1}$ is small enough, for all $n \ge 0$. (Note that $(L_{n+1})^{\tau-1} = (L_0 2^{n+1})^{\tau-1} \ll \exp(\chi^{n+1})$ for n large.) Furthermore, by (47), (37), and (48), estimate (45) holds.

We now prove in detail estimate (43) for $k = 1$. Differentiating equation (P_{n+1}) with respect to some coordinate λ of $(\delta, v_1) \in A_{n+1}$, we obtain

$$\mathscr{L}_{n+1}(\delta, v_1, w_{n+1})[\partial_\lambda h_{n+1}(\delta, v_1)] = -(\partial_\lambda U_{n+1})\big(\delta, v_1, h_{n+1}(\delta, v_1)\big), \tag{P'_{n+1}}$$

and therefore, by (45),

$$\|\partial_\lambda h_{n+1}\|_{\sigma_{n+1},s} \le \frac{C}{\gamma}(L_{n+1})^{\tau-1}\|(\partial_\lambda U_{n+1})(\delta, v_1, h_{n+1})\|_{\sigma_{n+1},s}. \tag{49}$$

To estimate the right-hand side of (49), first notice that since $w_n = w_n(\delta, v_1)$ solves

$$L_\omega w_n = \varepsilon P_n \Pi_W \Gamma(\delta, v_1, w_n), \quad \forall(\delta, v_1) \in A_n,$$

we have

$$U_{n+1}(\delta, v_1, h) = L_\omega h + \varepsilon\big(P_n \Pi_W \Gamma(\delta, v_1, w_n) - P_{n+1}\Pi_W \Gamma(\delta, v_1, w_n + h)\big).$$

Let us write

$$(\partial_\lambda U_{n+1})(\delta, v_1, h) = (\partial_\lambda U_{n+1})(\delta, v_1, 0) + r(\delta, v_1, h), \tag{50}$$

where

$$\begin{aligned}(\partial_\lambda U_{n+1})(\delta, v_1, 0) &= (P_n - P_{n+1})\Pi_W \partial_\lambda\big[\varepsilon(\delta)\Gamma\big(\delta, v_1, w_n(\delta, v_1)\big)\big] \\ &= -\varepsilon P_n^\perp P_{n+1}\Pi_W\big[(\partial_\lambda\Gamma)(\delta, v_1, w_n) + (\partial_w\Gamma)(\delta, v_1, w_n)[\partial_\lambda w_n]\big] \\ &\quad - \partial_\lambda\big(\varepsilon(\delta)\big) P_n^\perp P_{n+1}\Pi_W \Gamma(\delta, v_1, w_n)\end{aligned} \tag{51}$$

and

$$\begin{aligned} r(\delta, v_1, h) := {}& -\varepsilon P_{n+1}\Pi_W[(\partial_\lambda\Gamma)(\delta, v_1, w_n + h) - (\partial_\lambda\Gamma)(\delta, v_1, w_n)] \\ & - \varepsilon P_{n+1}\Pi_W[(\partial_w\Gamma)(\delta, v_1, w_n + h) - (\partial_w\Gamma)(\delta, v_1, w_n)][\partial_\lambda w_n] \\ & + \partial_\lambda(L_{\omega(\delta)}h) + \partial_\lambda\big(\varepsilon(\delta)\big)P_{n+1}\Pi_W\big(\Gamma(\delta, v_1, w_n) - \Gamma(\delta, v_1, w_n + h)\big),\end{aligned} \tag{52}$$

with $\partial_\lambda(L_{\omega(\delta)}h) = 0$, $\partial_\lambda(\varepsilon(\delta)) = 0$ if $\lambda \neq \delta$ and

$$\partial_\delta(L_{\omega(\delta)}h) = -2(p-1)\delta^{p-2}h_{tt}, \qquad \partial_\delta\big(\varepsilon(\delta)\big) = (p-1)\delta^{p-2}. \tag{53}$$

By (P1), (34), (52), and (53), for $h \in W^{(n+1)}$,

$$\|r(\delta, v_1, h)\|_{\sigma_{n+1},s} \le C|\varepsilon|\ \|h\|_{\sigma_{n+1},s}(1 + \|\partial_\lambda w_n\|_{\sigma_{n+1},s}) + CL_{n+1}^2\|h\|_{\sigma_{n+1},s}. \tag{54}$$

We now estimate $(\partial_\lambda U_{n+1})(\delta, v_1, 0)$. By (51) and properties (P2), (P1),

$$\begin{aligned}&\|(\partial_\lambda U_{n+1})(\delta, v_1, 0)\|_{\sigma_{n+1},s} \\ &\quad\le \exp(-L_n\gamma_n)\big[|\varepsilon|\|(\partial_\lambda\Gamma)(\delta, v_1, w_n) + (\partial_w\Gamma)(\delta, v_1, w_n)[\partial_\lambda w_n]\|_{\sigma_n,s} \\ &\qquad\qquad + \|\Gamma(\delta, v_1, w_n)\|_{\sigma_n,s}\big] \\ &\quad\le C\exp(-L_n\gamma_n)(1 + \|\partial_\lambda w_n\|_{\sigma_n,s}).\end{aligned} \tag{55}$$

Combining (49), (50), (54), (55), and the bound $\|h_{n+1}\|_{\sigma_{n+1},s} \le |\varepsilon|\gamma^{-1}\exp(-\chi^{n+1})$, we get

$$\begin{aligned}\|\partial_\lambda h_{n+1}\|_{\sigma_{n+1},s} &\le \frac{C}{\gamma}(L_{n+1})^{\tau+1}\Big(\frac{|\varepsilon|}{\gamma}\exp(-\chi^{n+1}) + \exp(-L_n\gamma_n)\Big)(1 + \|\partial_\lambda w_n\|_{\sigma_n,s}) \\ &\le C(\overline{\chi})\ \exp(-\overline{\chi}^{n+1})\Big(1 + \sum_{i=0}^{n}\|\partial_\lambda h_i\|_{\sigma_i,s}\Big)\end{aligned} \tag{56}$$

for any $\overline{\chi} \in (1, \chi)$. By (56), the sequence $a_n := \|\partial_\lambda h_n\|_{\sigma_n,s}$ satisfies

$$a_0 \le C \qquad \text{and} \qquad a_{n+1} \le C(\overline{\chi}) \exp(-\overline{\chi}^{n+1})(1 + a_0 + \cdots + a_n),$$

which implies (by induction)

$$\|\partial_\lambda h_n\|_{\sigma_n,s} = a_n \le K(\overline{\chi})^n \exp(-\overline{\chi}^n), \quad \forall n \ge 0,$$

provided that $K(\overline{\chi})$ has been chosen large enough. We can prove in the same way the general estimate (43), from which (44) follows. □

Since, by (43), $h_n(\delta, v_1) = O(\varepsilon\gamma^{-1} \exp(-\overline{\chi}^n))$, and the *nonresonant* set A_n is obtained at each step by deleting strips of size $O(\gamma/L_n^\tau)$, we can define (by interpolation, say) a C^∞-extension $\widetilde{w}(\delta, v_1)$ of $w(\delta, v_1)$ for all $(\delta, v_1) \in A_0$.

Let

$$\widetilde{A}_n := \left\{(\delta, v_1) \in A_n \,\middle|\, \operatorname{dist}\big((\delta, v_1), \partial A_n\big) \ge \frac{2\nu}{L_n^3}\right\} \subset A_n,$$

where $\nu\gamma^{-1} > 0$ is some small constant to be specified later (see Lemma 3.4).

LEMMA 3.3 (Whitney C^∞-extension $\widetilde{w}$ of w on A_0)
There exists a function $\widetilde{w}(\delta, v_1) \in C^\infty(A_0, W \cap X_{\overline{\sigma}/2,s})$ *satisfying*

$$\|\widetilde{w}(\delta, v_1)\|_{\overline{\sigma}/2,s} \le \frac{|\varepsilon|}{\gamma} C, \qquad \|D^k \widetilde{w}(\delta, v_1)\|_{\overline{\sigma}/2,s} \le \frac{C(k)}{\nu^k}, \quad \forall (\delta, v_1) \in A_0,\ \forall k \ge 1, \tag{57}$$

for some $C(k) > 0$*, such that*

$$\forall\, (\delta, v_1) \in \widetilde{A}_\infty := \bigcap_{n \ge 0} \widetilde{A}_n, \quad \widetilde{w}(\delta, v_1) \textit{ solves the } (P)\textit{-equation (25).}$$

Moreover, there exists a sequence $\widetilde{w}_n \in C^\infty(A_0, W^{(n)})$ *such that*

$$\widetilde{w}_n(\delta, v_1) = w_n(\delta, v_1), \quad \forall (\delta, v_1) \in \widetilde{A}_n, \tag{58}$$

and

$$\|\widetilde{w}(\delta, v_1) - \widetilde{w}_n(\delta, v_1)\|_{\overline{\sigma}/2,s} \le \frac{|\varepsilon| C}{\gamma} \exp(-\widetilde{\chi}^n), \tag{59}$$

$$\|D^k \widetilde{w}(\delta, v_1) - D^k \widetilde{w}_n(\delta, v_1)\|_{\overline{\sigma}/2,s} \le \frac{C(k)}{\nu^k} \exp(-\widetilde{\chi}^n), \quad \forall (\delta, v_1) \in A_0, \tag{60}$$

for some $\widetilde{\chi} \in (1, \overline{\chi})$.

Proof

Let $\varphi : \mathbf{R} \times V_1 \to \mathbf{R}_+$ be a C^∞-function supported in the open ball $B(0, 1)$ of center 0 and radius 1 with $\int_{\mathbf{R}\times V_1} \varphi \, d\mu = 1$. Here μ is the Borelian positive measure of $\mathbf{R} \times V_1$ defined by $\mu(E) = m(L^{-1}(E))$, where L is some automorphism from $\mathbf{R}^{N+1}$ to $\mathbf{R} \times V_1$ and m is the Lebesgue measure in $\mathbf{R}^{N+1}$.

Let $\varphi_n : \mathbf{R} \times V_1 \to \mathbf{R}_+$ be the *mollifier*

$$\varphi_n(\lambda) := \Big(\frac{L_n^3}{\nu}\Big)^{N+1} \varphi\Big(\frac{L_n^3}{\nu}\lambda\Big)$$

(here $\lambda := (\delta, v_1)$), which is a C^∞-function satisfying

$$\operatorname{supp}\varphi_n \subset B\Big(0, \frac{\nu}{L_n^3}\Big) \qquad \text{and} \qquad \int_{\mathbf{R}\times V_1} \varphi_n \, d\mu = 1. \tag{61}$$

Next, we define $\psi_n : \mathbf{R} \times V_1 \to \mathbf{R}_+$ as

$$\psi_n(\lambda) := (\varphi_n * \chi_{A_n^*})(\lambda) = \int_{\mathbf{R}\times V_1} \varphi_n(\lambda - \eta)\, \chi_{A_n^*}(\eta)\, d\mu(\eta),$$

where $\chi_{A_n^*}$ is the characteristic function of the set

$$A_n^* := \Big\{\lambda = (\delta, v_1) \in A_n \;\Big|\; \operatorname{dist}(\lambda, \partial A_n) \geq \frac{\nu}{L_n^3}\Big\} \subset A_n;$$

namely, $\chi_{A_n^*}(\lambda) := 1$ if $\lambda \in A_n^*$, and $\chi_{A_n^*}(\lambda) := 0$ if $\lambda \notin A_n^*$.

The function ψ_n is C^∞, and $\forall k \in \mathbf{N}$, $\forall \lambda \in \mathbf{R} \times V_1$,

$$\begin{aligned}
|D^k \psi_n(\lambda)| &= \Big| \int_{\mathbf{R}\times V_1} D^k \varphi_n(\lambda - \eta)\, \chi_{A_n^*}(\eta)\, d\mu(\eta)\Big| \\
&\leq \int_{\mathbf{R}\times V_1} \Big| \Big(\frac{L_n^3}{\nu}\Big)^k \Big(\frac{L_n^3}{\nu}\Big)^{N+1} (D^k\varphi)\Big(\frac{L_n^3}{\nu}(\lambda - \eta)\Big)\Big| \, d\mu(\eta) \\
&= \Big(\frac{L_n^3}{\nu}\Big)^k \int_{\mathbf{R}\times V_1} |D^k\varphi| \, d\mu = \Big(\frac{L_n^3}{\nu}\Big)^k C(k),
\end{aligned} \tag{62}$$

where $C(k) := \int_{\mathbf{R}\times V_1} |D^k\varphi| \, d\mu$. Furthermore, by (61) and the definition of A_n^* and $\widetilde{A}_n$,

$$0 \leq \psi_n(\lambda) \leq 1, \quad \operatorname{supp}\psi_n \subset \operatorname{int} A_n \qquad \text{and} \qquad \psi_n(\lambda) = 1 \quad \text{if } \lambda \in \widetilde{A}_n\,.$$

Finally, we can define $\widetilde{w}_n : A_0 \to W^{(n)}$ by

$$\widetilde{w}_0(\lambda) := w_0(\lambda), \qquad \widetilde{w}_{n+1}(\lambda) := \widetilde{w}_n(\lambda) + \widetilde{h}_{n+1}(\lambda) \in W^{(n+1)},$$

where

$$\widetilde{h}_{n+1}(\lambda) := \begin{cases} \psi_{n+1}(\lambda) h_{n+1}(\lambda) & \text{if } \lambda \in A_{n+1}, \\ 0 & \text{if } \lambda \notin A_{n+1}, \end{cases}$$

is in $C^\infty(A_0, W^{(n+1)})$ because supp $\psi_{n+1} \subset$ int A_{n+1} and, by Lemma 3.2, $h_{n+1} \in C^\infty(A_{n+1}, W^{(n+1)})$. Therefore we have

$$\widetilde{w}_n(\lambda) = \sum_{i=0}^{n} \widetilde{h}_i(\lambda), \quad \widetilde{w}_n \in C^\infty(A_0, W^{(n)}),$$

and (58) holds.

By the bounds (62) and (43), we obtain $\forall k \in \mathbf{N}$, $\forall \lambda \in A_0$, $\forall n \geq 0$,

$$\|\widetilde{h}_{n+1}(\lambda)\|_{\sigma_{n+1},s} \leq \frac{|\varepsilon| K}{\gamma} \exp(-\overline{\chi}^n),$$

$$\|D^k \widetilde{h}_{n+1}(\lambda)\|_{\sigma_{n+1},s} \leq C(k, \overline{\chi})^n \left(\frac{L_{n+1}^3}{\nu}\right)^k \exp(-\overline{\chi}^n) \leq \frac{K(k)}{\nu^k} \exp(-\widetilde{\chi}^n)$$

for some $1 < \widetilde{\chi} < \overline{\chi}$ and some positive constant $K(k)$ large enough. As a consequence, the sequence $(\widetilde{w}_n)$ (and all its derivatives) converges uniformly in A_0 for the norm $\| \ \|_{\overline{\sigma}/2,s}$ on W, to some function $\widetilde{w}(\delta, v_1) \in C^\infty(A_0, W \cap X_{\overline{\sigma}/2,s})$ which satisfies (57), (59), and (60).

Finally, note that if $\lambda \notin A_\infty := \bigcap_{n\geq 0} A_n$, then the series $\widetilde{w}(\lambda) = \sum_{n\geq 1} \widetilde{h}_n(\lambda)$ is a finite sum. On the other hand, if $\lambda \in \widetilde{A}_\infty := \bigcap_{n\geq 0} \widetilde{A}_n$, then $\widetilde{w}(\lambda) = w(\lambda)$ solves the (P)-equation (25). □

Remark 3.4

If $(\delta, v_1) \notin \widetilde{A}_\infty$, we claim that $\widetilde{w}(\delta, v_1)$ solves the (P)-equation up to exponentially small remainders. There exist $\alpha > 0$, $\delta_0(\gamma, \tau) > 0$ such that $\forall 0 < \delta \leq \delta_0(\gamma, \tau)$,

$$\left\| L_\omega \widetilde{w}(\delta, v_1) - \varepsilon \Pi_W \Gamma\big(\delta, v_1, \widetilde{w}(\delta, v_1)\big) \right\|_{\overline{\sigma}/4,s} \leq \frac{|\varepsilon|}{\gamma} \exp\left(-\frac{1}{\delta^\alpha}\right).$$

Since we do not use this property in the present article, we do not give here the proof.

3.3. Measure estimate

We now replace the set $\widetilde{A}_\infty$ with a smaller Cantor-like set B_∞ which has the advantage of being independent of the iteration steps. This is more convenient for the measure estimates required in Section 5. (This issue is discussed differently in [11].)

Define

$$B_n := \left\{(\delta, v_1) \in \widetilde{A}_0 \,\middle|\, \delta \in \Delta_n^{2\gamma,\tau}\big(v_1, \widetilde{w}(\delta, v_1)\big)\right\}, \tag{63}$$

where we have replaced γ with 2γ in the definition of $\Delta_n^{\gamma,\tau}$ (see Definition 3.3). Note that B_n does not depend on the approximate solution w_n but only on the fixed function $\widetilde{w}$.

LEMMA 3.4
If $\nu\gamma^{-1} > 0$ and $|\varepsilon|\gamma^{-1}$ are small enough, then

$$B_n \subset \widetilde{A}_n, \quad \forall n \geq 0.$$

Hence $B_\infty := \bigcap_{n\geq 1} B_n \subset \widetilde{A}_\infty \subset A_\infty$, and so if $(\delta, v_1) \in B_\infty$, then $\widetilde{w}(\delta, v_1)$ solves the (P)-equation (25).

Proof
We prove the lemma by induction. First, $B_0 \subset \widetilde{A}_0$. Suppose next that $B_n \subset \widetilde{A}_n$ holds. In order to prove that $B_{n+1} \subset \widetilde{A}_{n+1}$, take any $(\delta, v_1) \in B_{n+1}$. We have to justify that the ball $B((\delta, v_1), 2\nu/L_{n+1}^3) \subset A_{n+1}$.

First, since $B_{n+1} \subset B_n \subset \widetilde{A}_n$, $(\delta, v_1) \in \widetilde{A}_n$. Hence, since $L_{n+1} > L_n$, $B((\delta, v_1), 2\nu/L_{n+1}^3) \subset A_n$.

Let $(\delta', v_1') \in B((\delta, v_1), 2\nu/L_{n+1}^3)$. Since $(\delta, v_1) \in \widetilde{A}_n$, we have $\widetilde{w}_n(\delta, v_1) = w_n(\delta, v_1)$. Moreover, by (44), $\|Dw_n\|_{\overline{\sigma}/2,s} \leq C$. By (59), we can derive

$$\begin{aligned}
&\|w_n(\delta', v_1') - \widetilde{w}(\delta, v_1)\|_{\overline{\sigma}/2,s} \\
&\quad \leq \|w_n(\delta', v_1') - w_n(\delta, v_1)\|_{\overline{\sigma}/2,s} + \|w_n(\delta, v_1) - \widetilde{w}(\delta, v_1)\|_{\overline{\sigma}/2,s} \\
&\quad \leq \frac{2\nu C}{L_{n+1}^3} + \frac{C|\varepsilon|}{\gamma}\exp(-\widetilde{\chi}^n).
\end{aligned}$$

Hence, by (63), setting $\omega' := \sqrt{1 + 2(\delta')^{p-1}}$ and $\varepsilon' := (\delta')^{p-1}$ (for simplicity of notation, suppose that $s^* = 1$),

$$\begin{aligned}
&\left|\omega' l - j - \varepsilon' \frac{M(\delta', v_1', w_n(\delta', v_1'))}{2j}\right| \\
&\geq \left|\omega l - j - \varepsilon \frac{M(\delta, v_1, \widetilde{w}(\delta, v_1))}{2j}\right| - l\frac{C\nu}{L_{n+1}^3} - C\frac{|\varepsilon|\nu}{L_{n+1}^3} - C\frac{|\varepsilon|^2}{\gamma}\exp(-\widetilde{\chi}^n) \\
&\geq \frac{2\gamma}{(l+j)^\tau} - \frac{C\nu}{L_{n+1}^2} - C\frac{|\varepsilon|^2}{\gamma}\exp(-\widetilde{\chi}^n) \geq \frac{\gamma}{(l+j)^\tau}
\end{aligned}$$

for all $1/3|\varepsilon| < l < L_{n+1}, l \neq j, j \leq 2L_{n+1}$, whenever

$$\frac{\gamma}{(3L_{n+1})^\tau} \geq C\Big(\frac{\nu}{L_{n+1}^2} + \frac{|\varepsilon|^2}{\gamma}\exp(-\widetilde{\chi}^n)\Big). \tag{64}$$

Formula (64) holds true, for $|\varepsilon|\gamma^{-1}$ and $\nu\gamma^{-1}$ small, for all $n \geq 0$, because $\tau < 2$ and $\lim_{n\to\infty} L_{n+1}^{\tau} \exp(-\widetilde{\chi}^n) = 0$. It results in $B((\delta, v_1), 2\nu/L_{n+1}^3)) \subset A_{n+1}$. □

Up to now, we have not justified the fact that

$$B_\infty \subset \widetilde{A}_\infty \subset A_\infty \tag{65}$$

are not reduced to $\{\delta = 0\} \times B(2R, V_1)$. It is a consequence of the following result, which is applied in Section 5.

PROPOSITION 3.2 (Measure estimate of B_∞)
Let $\mathscr{V}_1 : [0, \delta_0) \to V_1$ be a C^1-function. Then

$$\lim_{\eta\to 0^+} \frac{\operatorname{meas}\{\delta \in [0, \eta) \mid (\delta, \mathscr{V}_1(\delta)) \in B_\infty\}}{\eta} = 1. \tag{66}$$

Proof
Let $0 < \eta < \delta_0$. Define

$$\mathscr{C}_{\mathscr{V}_1,\eta} := \left\{\delta \in (0, \eta) \,\middle|\, \big(\delta, \mathscr{V}_1(\delta)\big) \in B_\infty\right\} \qquad \text{and} \qquad \mathscr{D}_{\mathscr{V}_1,\eta} := (0, \eta)\backslash\mathscr{C}_{\mathscr{V}_1,\eta}.$$

By the definition $B_\infty := \bigcap_{n\geq 1} B_n$ (see also the expression of B_∞ in the statement of Theorem 3.1, where for simplicity of notation, we suppose that $s^* = 1$),

$$\mathscr{D}_{\mathscr{V}_1,\eta} = \Big\{\delta \in (0, \eta) \,\Big|\, \Big|\omega(\delta)l - j - \frac{\delta^{p-1}m(\delta)}{2j}\Big| < \frac{2\gamma}{(l+j)^\tau}$$
$$\text{or} \quad \Big|\omega(\delta)l - j\Big| < \frac{2\gamma}{(l+j)^\tau} \text{ for some } l, j > \frac{1}{3\delta^{p-1}},\ l \neq j \Big\},$$

where $m(\delta) := M\big(\delta, \mathscr{V}_1(\delta), \widetilde{w}(\delta, \mathscr{V}_1(\delta))\big)$ is a function in $C^1([0, \delta_0), \mathbf{R})$ since $\widetilde{w}(\cdot, \cdot)$ is in $C^\infty(A_0, W \cap X_{\overline{\sigma}/2,s}))$ and $\mathscr{V}_1$ is C^1. This implies, in particular,

$$|m(\delta)| + |m'(\delta)| \leq C, \quad \forall \delta \in \Big[0, \frac{\delta_0}{2}\Big] \tag{67}$$

for some positive constant C.

We claim that for any interval $[\delta_1/2, \delta_1] \subset [0, \eta] \subset [0, \delta_0/2]$, the following measure estimate holds:

$$\operatorname{meas}\Big(\mathscr{D}_{\mathscr{V}_1,\eta} \cap \Big[\frac{\delta_1}{2}, \delta_1\Big]\Big) \leq K_1(\tau)\gamma\eta^{(p-1)(\tau-1)} \operatorname{meas}\Big(\Big[\frac{\delta_1}{2}, \delta_1\Big]\Big) \tag{68}$$

for some constant $K_1(\tau) > 0$.

Before proving (68), we show how to conclude the proof of the proposition. Writing $(0, \eta] = \bigcup_{n\geq 0} [\eta/2^{n+1}, \eta/2^n]$ and applying the measure estimate (68) to any interval $[\delta_1/2, \delta_1] = [\eta/2^{n+1}, \eta/2^n]$, we get

$$\text{meas}(\mathscr{D}_{\mathscr{V}_{1,\eta}} \cap [0, \eta]) \le K_1(\tau)\gamma\eta^{(p-1)(\tau-1)}\eta,$$

whence $\lim_{\eta\to 0^+} \text{meas}\,(\mathscr{C}_{\mathscr{V}_{1,\eta}} \cap (0, \eta))/\eta = 1$, proving the proposition.

We now prove (68). We have

$$\mathscr{D}_{\mathscr{V}_{1,\eta}} \bigcap \left[\frac{\delta_1}{2}, \delta_1\right] \subset \bigcup_{(l,j)\in I_R} \mathscr{R}_{l,j}(\delta_1), \tag{69}$$

where

$$\mathscr{R}_{l,j}(\delta_1) := \left\{\delta \in \left[\frac{\delta_1}{2}, \delta_1\right] \,\Big|\, \left|\omega(\delta)l - j - \frac{\delta^{p-1}m(\delta)}{2j}\right| < \frac{2\gamma}{(l+j)^\tau} \right.$$
$$\left. \text{or } \left|\omega(\delta)l - j\right| < \frac{2\gamma}{(l+j)^\tau}\right\}$$

and

$$I_R := \left\{(l, j) \,\Big|\, l > \frac{1}{3\delta_1^{p-1}},\ l \ne j,\ \frac{j}{l} \in [1 - c_0\delta_1^{p-1}, 1 + c_0\delta_1^{p-1}]\right\}.$$

(Indeed, note that $\mathscr{R}_{j,l}(\delta_1) = \emptyset$ unless $j/l \in [1 - c_0\delta_1^{p-1}, 1 + c_0\delta_1^{p-1}]$ for some constant $c_0 > 0$ large enough.)

Next, let us prove that

$$\text{meas}\big(\mathscr{R}_{lj}(\delta_1)\big) = O\Big(\frac{\gamma}{l^{\tau+1}\delta_1^{p-2}}\Big). \tag{70}$$

Define $f_{lj}(\delta) := \omega(\delta)l - j - (\delta^{p-1}m(\delta)/2j)$ and $\mathscr{S}_{j,l}(\delta_1) := \{\delta \in [\delta_1/2, \delta_1] : |f_{l,j}(\delta)| < 2\gamma/(l+j)^\tau\}$. Provided that δ_0 has been chosen small enough (recall that $j, l \ge 1/3\delta_0^{p-1}$),

$$|\partial_\delta f_{lj}(\delta)| = \Big|\frac{l(p-1)\delta^{p-2}}{\sqrt{1+2\delta^{p-1}}} - \frac{(p-1)\delta^{p-2}m(\delta)}{2j} - \frac{\delta^{p-1}m'(\delta)}{2j}\Big|$$
$$\ge \frac{(p-1)\delta^{p-2}}{2}\Big(l - \frac{C}{j}\Big) \ge \frac{(p-1)\delta^{p-2}l}{4},$$

and therefore $|\partial_\delta f_{lj}(\delta)| \ge (p-1)\delta_1^{p-2}l/2^p$ for any $\delta \in [\delta_1/2, \delta_1]$. This implies

$$\text{meas}\big(\mathscr{S}_{lj}(\delta_1)\big) \le \frac{4\gamma}{(l+j)^\tau} \times \Big(\min_{\delta\in[\delta_1/2,\delta_1]} |\partial_\delta f_{lj}(\delta)|\Big)^{-1}$$
$$\le \frac{4\gamma}{(l+j)^\tau} \times \frac{2^p}{(p-1)l\delta_1^{p-2}} = O\Big(\frac{\gamma}{l^{\tau+1}\delta_1^{p-2}}\Big).$$

Similarly, we can prove

$$\mathrm{meas}\Big(\Big\{\delta\in\Big[\frac{\delta_1}{2},\delta_1\Big]\ :\ |\omega(\delta)l-j|<\frac{2\gamma}{(l+j)^\tau}\Big\}\Big)=O\Big(\frac{\gamma}{l^{\tau+1}\delta_1^{p-2}}\Big),$$

and the measure estimate (70) follows.

Now, by (69) and (70) and since, for a given l, the number of j for which $(l,\ j)\in I_R$ is $O(\delta_1^{p-1}l)$,

$$\mathrm{meas}\Big(\mathscr{D}_{\mathscr{V}_1,\eta}\cap\Big[\frac{\delta_1}{2},\delta_1\Big]\Big)\le\sum_{(l,j)\in I_R}\mathrm{meas}\big(\mathscr{R}_{j,l}(\delta_1)\big)\le C\sum_{l\ge 1/3\delta_1^{p-1}}\delta_1^{p-1}l\times\frac{\gamma}{l^{\tau+1}\delta_1^{p-2}}$$
$$\le K_2(\tau)\gamma\delta_1^{1+(p-1)(\tau-1)},$$

whence we obtain (68) since $0<\delta_1<\eta$. □

We summarize the main result of this section as follows.

THEOREM 3.1 (Solution of the (P)-equation)
For $\delta_0:=\delta_0(\gamma,\tau)>0$ small enough, there exist a C^∞-function $\widetilde{w}\ :\ A_0:=\{(\delta,v_1)\mid \delta\in[0,\delta_0),\ \ \|v_1\|_{0,s}\le 2R\}\to W\cap X_{\overline{\sigma}/2,s}$ satisfying (57), and the large *(see (66)) Cantor set*

$$B_\infty:=\Big\{(\delta,v_1)\in A_0\ :\ \Big|\omega(\delta)l-j-s^*\delta^{p-1}\frac{M(\delta,v_1,\widetilde{w}(\delta,v_1))}{2j}\Big|\ge\frac{2\gamma}{(l+j)^\tau},$$
$$\Big|\omega(\delta)l-j\Big|\ge\frac{2\gamma}{(l+j)^\tau},\ \forall l\ge\frac{1}{3\delta^{p-1}},\ l\ne j\ \Big\}\subset A_0,$$

where $\omega(\delta)=\sqrt{1+2s^*\delta^{p-1}}$ and $M(\delta,v_1,w)$ is defined in Definition 3.1 such that

$$\forall(\delta,v_1)\in B_\infty,\quad \widetilde{w}(\delta,v_1)\ \text{solves the}\ \ (P)\text{-equation (25)}.$$

4. Analysis of the linearized problem: Proof of (P3)

We prove in this section the key property (P3) on the inversion of the linear operator $\mathscr{L}_n(\delta,v_1,w)$ defined in (28).

Throughout this section we use the notation

$$F_k:=\Big\{f\in H_0^1\big((0,\pi);\mathbf{R}\big)\ \Big|\ \int_0^\pi f(x)\sin(kx)\,dx=0\Big\}=\langle\sin(kx)\rangle^\perp,$$

whence the space W, defined in (6), is written as

$$W=\Big\{h=\sum_{k\in\mathbf{Z}}\exp(\mathrm{i}kt)h_k\in X_{0,s}\ \Big|\ h_k=h_{-k},\ h_k\in F_k,\ \forall k\in\mathbf{Z}\Big\}$$

and the corresponding projector $\Pi_W : X_{\sigma,s} \to W$ is

$$(\Pi_W h)(t,x) = \sum_{k\in\mathbf{Z}} \exp(\mathrm{i}kt)(\pi_k h_k)(x), \tag{71}$$

where $\pi_k : H_0^1((0,\pi);\mathbf{R}) \to F_k := \langle \sin(kx)\rangle^{\perp}$ is the L^2-orthogonal projector onto F_k,

$$(\pi_k f)(x) := f(x) - \Big(\frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx\Big)\sin(kx).$$

Note that $\pi_{-k} = \pi_k$. Hence, since $h_k = h_{-k}$, $\pi_k h_k = \pi_{-k} h_{-k}$.

4.1. Decomposition of $\mathscr{L}_n(\delta, v_1, w)$

Recalling (26), the operator $\mathscr{L}_n(\delta, v_1, w) : D(\mathscr{L}_n) \subset W^{(n)} \to W^{(n)}$ is written as

$$\begin{aligned} &\mathscr{L}_n(\delta, v_1, w)[h] \\ &\quad := L_\omega h - \varepsilon P_n \Pi_W D_w \Gamma(\delta, v_1, w)[h] \\ &\quad = L_\omega h - \varepsilon P_n \Pi_W \big(\partial_u g(\delta, x, v_1 + w + v_2(\delta, v_1, w))(h + \partial_w v_2(\delta, v_1, w)[h])\big) \\ &\quad = L_\omega h - \varepsilon P_n \Pi_W \big(a(t,x)\, h\big) - \varepsilon P_n \Pi_W \big(a(t,x)\, \partial_w v_2(\delta, v_1, w)[h]\big), \end{aligned} \tag{72}$$

where, for brevity, we have set

$$a(t,x) := \partial_u g\big(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x)\big). \tag{73}$$

In order to invert $\mathscr{L}_n$, it is convenient to perform a Fourier expansion and represent the operator $\mathscr{L}_n$ as a matrix, distinguishing a *diagonal* matrix D and an *off-diagonal Toepliz* matrix. The main difference with respect to the analogue procedure of Craig and Wayne [11] and Bourgain [7] is that we develop $\mathscr{L}_n$ only in *time*-Fourier basis and not also in the spatial fixed basis formed by the eigenvectors $\sin(jx)$ of the linear operator $-\partial_{xx}$. The reason is that this is more convenient to deal with nonlinearities $f(x,u)$ with finite regularity in x and without oddness assumptions. Each diagonal element D_k is a differential operator acting on functions of x. Next, using Sturm-Liouville theory, we diagonalize each D_k in a suitable basis of eigenfunctions close, but different, from $\sin jx$ (see Lemma 4.1, Corollary 4.1).

Performing a time-Fourier expansion, the operator $L_\omega := -\omega^2\partial_{tt} + \partial_{xx}$ is diagonal since

$$L_\omega\Big(\sum_{|k|\le L_n} \exp(\mathrm{i}kt)h_k\Big) = \sum_{|k|\le L_n} \exp(\mathrm{i}kt)(\omega^2 k^2 + \partial_{xx})h_k. \tag{74}$$

The operator $h \to P_n \Pi_W (a(t,x)\, h)$ is the composition of the multiplication operator for the function $a(t,x) = \sum_{l\in\mathbf{Z}} \exp(\mathrm{i}lt) a_l(x)$ with the projectors Π_W and P_n. As usual, in Fourier expansion, the multiplication operator is described by a *Toepliz matrix*

$$a(t,x)\,h(t,x) = \sum_{|k|\le L_n, l\in\mathbf{Z}} \exp(\mathrm{i}lt) a_{l-k}(x) h_k(x)$$

and, recalling (71) and (27),

$$\begin{aligned} P_n\Pi_W\big(a(t,x)\,h\big) &= \sum_{|k|,|l|\le L_n} \exp(\mathrm{i}lt)\pi_l\big(a_{l-k}(x)h_k\big) \\ &= \sum_{|k|\le L_n} \exp(\mathrm{i}kt)\pi_k\big(a_0(x)h_k\big) + \sum_{|k|,|l|\le L_n, k\ne l} \exp(\mathrm{i}lt)\pi_l(a_{l-k}h_k), \end{aligned} \quad (75)$$

where we have distinguished the *diagonal* term

$$\sum_{|k|\le L_n} \exp(\mathrm{i}kt)\pi_k\big(a_0(x)h_k\big) = P_n\Pi_W\big(a_0(x)\,h\big), \quad (76)$$

with $a_0(x) := (1/(2\pi))\int_0^{2\pi} a(t,x)\,dt$, from the *off-diagonal Toepliz* term

$$\sum_{|k|,|l|\le L_n, k\ne l} \exp(\mathrm{i}lt)\pi_l(a_{l-k}h_k) = P_n\Pi_W\big(\overline{a}(t,x)\,h\big), \quad (77)$$

where

$$\overline{a}(t,x) := a(t,x) - a_0(x)$$

has zero time-average.

By (72), (75), (76), and (77), we can decompose

$$\mathscr{L}_n(\delta, v_1, w) = D - \mathscr{M}_1 - \mathscr{M}_2,$$

where D, $\mathscr{M}_1$, $\mathscr{M}_2$ are the linear operators

$$\begin{cases} Dh := L_\omega h - \varepsilon P_n\Pi_W\big(a_0(x)\,h\big), \\ \mathscr{M}_1 h := \varepsilon P_n\Pi_W\big(\overline{a}(t,x)\,h\big), \\ \mathscr{M}_2 h := \varepsilon P_n\Pi_W\big(a(t,x)\,\partial_w v_2[h]\big). \end{cases} \quad (78)$$

To invert $\mathscr{L}_n$, we first (step 1) prove that, assuming the *first-order Melnikov nonresonance condition* $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$ (see Definition 3.3), the diagonal (in time) linear operator D is invertible (see Corollary 4.2). Next (step 2), we prove that the *off-diagonal Toepliz* operators $\mathscr{M}_1$ (see Lemma 4.8) and $\mathscr{M}_2$ (see Lemma 4.9) are small enough with respect to D, yielding the invertibility of the whole $\mathscr{L}_n$. (Note that we do not decompose the term $\mathscr{M}_2$ in a *diagonal* and *off-diagonal term*.) More precisely, the crucial bounds of Lemma 4.5 enable us to prove via Lemma 4.6 that the operator

$|D|^{-1/2}\mathscr{M}_1|D|^{-1/2}$ has a small norm, whereas the norm of $|D|^{-1/2}\mathscr{M}_2|D|^{-1/2}$ is controlled thanks to the regularizing properties of the map v_2.

4.2. *Step 1: Inversion of D*

The first aim is to diagonalize (both in time and space) the linear operator D (see Corollary 4.1).

By (74) and (76), the operator D is yet diagonal in time-Fourier basis, and $\forall\, h \in W^{(n)}$, the kth time Fourier coefficient of Dh is

$$(Dh)_k = (\omega^2 k^2 + \partial_{xx})h_k - \varepsilon\pi_k\big(a_0(x)h_k\big) \equiv D_k h_k,$$

where $D_k : \mathscr{D}(D_k) \subset F_k \to F_k$ is the operator

$$D_k u = \omega^2 k^2 u - S_k u \qquad \text{and} \qquad S_k u := -\partial_{xx} u + \varepsilon\pi_k\big(a_0(x)\, u\big).$$

Note that $S_k = S_{-k}$.

We now have to diagonalize (in space) each Sturm-Liouville type operator S_k and study its spectral properties.

In Lemma 4.1 we find a basis of eigenfunctions $v_{k,j}$ of $S_k : \mathscr{D}(S_k) \subset F_k \to F_k$ which are orthonormal for the scalar product of F_k,

$$\langle u, v\rangle_\varepsilon := \int_0^\pi u_x v_x + \varepsilon a_0(x) uv\, dx.$$

For $|\varepsilon||a_0|_\infty < 1$, $\langle\ ,\ \rangle_\varepsilon$ actually defines a scalar product on $F_k \subset H_0^1((0,\pi);\mathbf{R})$, and its associated norm is equivalent to the H^1-norm defined by $\|u\|_{H^1}^2 := \int_0^\pi u_x^2(x)\, dx$ since

$$\|u\|_{H^1}^2(1 - |\varepsilon|\, |a_0|_\infty) \le \|u\|_\varepsilon^2 \le \|u\|_{H^1}^2(1 + |\varepsilon|\, |a_0|_\infty), \quad \forall u \in F_k. \tag{79}$$

Formula (79) follows from* $\int_0^\pi u(x)^2\, dx \le \int_0^\pi u_x(x)^2\, dx$, $\forall u \in H_0^1(0,\pi)$, and

$$\Big|\int_0^\pi \varepsilon a_0(x) u^2\, dx\Big| \le |\varepsilon|\, |a_0|_\infty \int_0^\pi u^2\, dx.$$

LEMMA 4.1 (Sturm-Liouville)
The operator $S_k : \mathscr{D}(S_k) \subset F_k \to F_k$ possesses a $\langle\, ,\rangle_\varepsilon$-orthonormal basis $(v_{k,j})_{j\ge 1, j\ne|k|}$ of eigenvectors with positive, simple eigenvalues

$$0 < \lambda_{k,1} < \cdots < \lambda_{k,|k|-1} < \lambda_{k,|k|+1} < \cdots < \lambda_{k,j} < \cdots \quad \text{with} \ \lim_{j\to\infty} \lambda_{k,j} = +\infty$$

and $\lambda_{k,j} = \lambda_{-k,j}$, $v_{-k,j} = v_{k,j}$.

Moreover, $(v_{k,j})_{j\ge 1, j\ne|k|}$ is an orthogonal basis also for the L^2-scalar product in F_k.

*This is because the least eigenvalue of $-\partial_{xx}$ with Dirichlet boundary conditions on $(0,\pi)$ is 1.

The asymptotic expansion as $j \to +\infty$ *of the eigenfunctions* $\varphi_{k,j} := v_{k,j} / \|v_{k,j}\|_{L^2}$ *of* S_k *and its eigenvalues* $\lambda_{k,j}$ *is*

$$\left|\varphi_{k,j} - \sqrt{\frac{2}{\pi}}\sin(jx)\right|_{L^2} = O\Big(\frac{\varepsilon|a_0|_\infty}{j}\Big)$$

and

$$\lambda_{k,j} = \lambda_{k,j}(\delta, v_1, w) = j^2 + \varepsilon M(\delta, v_1, w) + O\Big(\frac{\varepsilon\|a_0\|_{H^1}}{j}\Big), \tag{80}$$

where $M(\delta, v_1, w)$*, introduced in Definition 3.1, is the mean value of* $a_0(x)$ *on* $(0, \pi)$.

The proof of this lemma is in the appendix. We note that we do not directly apply some known result for Sturm-Liouville operators because of the projection π_k.

By Lemma 4.1, each linear operator $D_k : \mathscr{D}(D_k) \subset F_k \to F_k$ possesses a $\langle\, ,\rangle_\varepsilon$-orthonormal basis $(v_{k,j})_{j\geq 1, j\neq |k|}$ of real eigenvectors with real eigenvalues $(\omega^2 k^2 - \lambda_{k,j})_{j\geq 1, j\neq |k|}$.

As a consequence, we have the following.

COROLLARY 4.1 (Diagonalization of D)
The operator D (*acting in* $W^{(n)}$) *is the diagonal operator* $\mathrm{diag}\{\omega^2k^2 - \lambda_{k,j}\}$ *in the basis* $\{\cos(kt)\varphi_{k,j}\ ;\ k \geq 0,\ j \geq 1,\ j \neq k\}$ *of* $W^{(n)}$.

By Lemma 4.1,

$$\min_{|k|\leq L_n} |\omega^2 k^2 - \lambda_{k,j}| \to +\infty \quad \text{as } j \to +\infty,$$

and so, by Corollary 4.1, the linear operator D is invertible if and only if all its eigenvalues $\{\omega^2k^2 - \lambda_{k,j}\}_{|k|\leq L_n, j\geq 1, j\neq |k|}$ are different from zero.

In this case, we can define D^{-1} as well as $|D|^{-1/2} : W^{(n)} \to W^{(n)}$ by

$$|D|^{-1/2}h := \sum_{|k|\leq L_n} \exp(\mathrm{i}kt)|D_k|^{-1/2}h_k, \quad \forall h = \sum_{|k|\leq L_n} \exp(\mathrm{i}kt)h_k,$$

where $|D_k|^{-1/2} : F_k \to F_k$ is the diagonal operator defined by

$$|D_k|^{-1/2}v_{k,j} := \frac{v_{k,j}}{\sqrt{|\omega^2k^2 - \lambda_{k,j}|}}, \quad \forall j \geq 1,\ j \neq |k|. \tag{81}$$

The "small denominators" problem (i) is that some of the eigenvalues of D, $\omega^2k^2 - \lambda_{k,j}$, can become arbitrarily small for $(k, j) \in \mathbf{Z}^2$ sufficiently large, and therefore the norm of $|D|^{-1/2}$ can become arbitrarily large as $L_n \to \infty$.

In order to quantify this phenomenon, we define for all $|k| \le L_n$,

$$\alpha_k := \min_{j \neq |k|} |\omega^2 k^2 - \lambda_{k,j}|. \tag{82}$$

Note that $\alpha_{-k} = \alpha_k$.

LEMMA 4.2
Suppose that $\alpha_k \neq 0$. Then D_k is invertible and, for ε small enough,

$$\| |D_k|^{-1/2} u \|_{H^1} \le \frac{2}{\sqrt{\alpha_k}} \|u\|_{H^1}. \tag{83}$$

Proof
For any $u = \sum_{j \neq |k|} u_j v_{k,j} \in F_k$, by (81), and using the fact that $(v_{k,j})_{j \neq |k|}$ is an orthonormal basis for the $\langle\ ,\ \rangle_\varepsilon$ scalar product on F_k,

$$\begin{aligned} \| |D_k|^{-1/2} u \|_\varepsilon^2 &= \Big\| \sum_{j \neq |k|} \frac{u_j\, v_{k,j}}{\sqrt{|\omega^2 k^2 - \lambda_{k,j}|}} \Big\|_\varepsilon^2 \\ &= \sum_{j \neq |k|} \frac{|u_j|^2}{|\omega^2 k^2 - \lambda_{k,j}|} \le \frac{1}{\alpha_k} \sum_{j \neq |k|} |u_j|^2 = \frac{\|u\|_\varepsilon^2}{\alpha_k}. \end{aligned}$$

Hence, since by (79) the norms $\| \cdot \|_\varepsilon$ and $\| \cdot \|_{H^1}$ are equivalent, (83) follows (for ε small enough). □

The condition $\alpha_k \neq 0$, $\forall |k| \le L_n$, depends very sensitively on the parameters (δ, v_1). Assuming the *first-order Melnikov nonresonance condition* $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$ (see Definition 3.3) with $\tau \in (1, 2)$, we obtain, in Lemma 4.3, a lower bound of the form $c\gamma/|k|^{\tau-1}$ for the moduli of the eigenvalues of D_k (namely, $\alpha_k \ge c\gamma/|k|^{\tau-1}$) and, therefore, in Corollary 4.2, sufficiently good estimates for the inverse of D.

LEMMA 4.3 (Lower bound for the eigenvalues of D)
There is $c > 0$ such that if $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ and δ_0 is small enough (depending on γ), then

$$\alpha_k := \min_{j \ge 1, j \neq |k|} |\omega^2 k^2 - \lambda_{k,j}| \ge \frac{c\,\gamma}{|k|^{\tau-1}} > 0, \quad \forall\, 0 < |k| \le L_n. \tag{84}$$

Moreover, $\alpha_0 \ge 1/2$.

Proof
Since $\alpha_{-k} = \alpha_k$, it is sufficient to consider $k \geq 0$. By the asymptotic expansion (80) for the eigenvalues $\lambda_{k,j}$, using that $\|a_0\|_{H^1}$, $|M(\delta, v_1, w)| \leq C$,

$$\begin{aligned}
|\omega^2 k^2 - \lambda_{k,j}| &= \Big|\omega^2 k^2 - j^2 - \varepsilon M(\delta, v_1, w) + O\Big(\frac{\varepsilon \|a_0\|_{H^1}}{j}\Big)\Big| \\
&= \Big|\Big(\omega k - \sqrt{j^2 + \varepsilon M(\delta, v_1, w)}\Big)\Big(\omega k + \sqrt{j^2 + \varepsilon M(\delta, v_1, w)}\Big) + O\Big(\frac{|\varepsilon|}{j}\Big)\Big| \\
&\geq \Big|\omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j} + O\Big(\frac{\varepsilon^2}{j^3}\Big)\Big|\omega k - C\frac{|\varepsilon|}{j} \\
&\geq \Big|\omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j}\Big| \, \omega k - C'\Big(\frac{\varepsilon^2 k}{j^3} + \frac{|\varepsilon|}{j}\Big) \\
&\geq \frac{\gamma \omega k}{(k + j)^\tau} - C\Big(\frac{\varepsilon^2 k}{j^3} + \frac{|\varepsilon|}{j}\Big)
\end{aligned} \tag{85}$$

since $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$. If $\alpha_k := \min_{j \geq 1, j \neq k} |\omega^2 k^2 - \lambda_{k,j}|$ is attained at $j = j(k)$ (i.e., $\alpha_k = |\omega^2 k^2 - \lambda_{k,j}|$), then $|\omega k - j| \leq 1$ (provided that $|\varepsilon|$ is small enough). Therefore, using that $1 < \tau < 2$ and $|\omega - 1| \leq 2|\varepsilon|$, we can derive (84) from (85), for $|\varepsilon|$ small enough. □

COROLLARY 4.2 (Estimate of $|D|^{-1/2}$)
If $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ and δ_0 is small enough, then $D : \mathscr{D}(D) \subset W^{(n)} \to W^{(n)}$ is invertible and, $\forall s' \geq 0$,

$$\||D|^{-1/2} h\|_{\sigma,s'} \leq \frac{C}{\sqrt{\gamma}} \|h\|_{\sigma,s'+(\tau-1)/2}, \quad \forall h \in W^{(n)}. \tag{86}$$

Proof
Since $|D|^{-1/2} h := \sum_{|k| \leq L_n} \exp(\mathrm{i}kt) |D_k|^{-1/2} h_k$, we get, using (83) and (84),

$$\begin{aligned}
\||D|^{-1/2} h\|^2_{\sigma,s'} &= \sum_{|k| \leq L_n} \exp(2\sigma|k|)(1 + k^{2s'}) \||D_k|^{-1/2} h_k\|^2_{H^1} \\
&\leq \sum_{|k| \leq L_n} \exp(2\sigma|k|)(1 + k^{2s'}) \frac{4}{\alpha_k} \|h_k\|^2_{H^1} \\
&\leq 8\|h_0\|^2_{H^1} + C \sum_{0 < |k| \leq L_n} \exp(2\sigma|k|)(1 + k^{2s'}) \frac{|k|^{(\tau-1)}}{\gamma} \|h_k\|^2_{H^1} \\
&\leq \frac{C'}{\gamma} \|h\|^2_{\sigma,s'+(\tau-1)/2},
\end{aligned}$$

proving (86). □

4.3. Step 2: Inversion of $\mathscr{L}_n$

To show the invertibility of $\mathscr{L}_n : W^{(n)} \to W^{(n)}$, it is a convenient device to write

$$\mathscr{L}_n = D - \mathscr{M}_1 - \mathscr{M}_2 = |D|^{1/2}(U - \mathscr{R}_1 - \mathscr{R}_2)|D|^{1/2},$$

where

$$U := |D|^{-1/2} D |D|^{-1/2} = |D|^{-1} D$$

and

$$\mathscr{R}_i := |D|^{-1/2} \mathscr{M}_i |D|^{-1/2}, \quad i = 1, 2.$$

We prove the invertibility of $U - \mathscr{R}_1 - \mathscr{R}_2$ showing that, for ε small enough, $\mathscr{R}_1$ and $\mathscr{R}_2$ are small perturbations of U.

LEMMA 4.4 (Estimate of $\|U^{-1}\|$)
$U : W^{(n)} \to W^{(n)}$ is an invertible operator, and its inverse U^{-1} satisfies, $\forall s' \geq 0$,

$$\|U^{-1} h\|_{\sigma,s'} = \|h\|_{\sigma,s'} \big(1 + O(\varepsilon \|a_0\|_{H^1})\big), \quad \forall\, h \in W^{(n)}. \tag{87}$$

Proof

Since $U_k := |D_k|^{-1} D_k : F_k \to F_k$ is orthogonal for the $\langle\, , \rangle_\varepsilon$ scalar product, it is invertible and $\forall u \in F_k$, $\|U_k^{-1} u\|_\varepsilon = \|u\|_\varepsilon$. Hence, by (79),

$$\forall u \in F_k, \quad \|U_k^{-1} u\|_{H^1} = \|u\|_\varepsilon \big(1 + O(\varepsilon \|a_0\|_{H^1})\big).$$

Therefore $U = |D|^{-1} D$, defined by $(Uh)_k = U_k h_k$, $\forall\, |k| \leq L_n$, U is invertible, $(U^{-1} h)_k = U_k^{-1} h_k$, and (87) holds. □

The estimate of the *off-diagonal* operator $\mathscr{R}_1 : W^{(n)} \to W^{(n)}$ requires a careful analysis of the "small divisors" and the use of the *first-order Melnikov nonresonance condition* $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$ (see Definition 3.3). For clarity, we state such a property separately.

LEMMA 4.5 (Analysis of the "small divisors")
Let $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ with δ_0 small. There exists $C > 0$ such that, $\forall l \neq k$,

$$\frac{1}{\alpha_k \alpha_l} \leq C \frac{|k - l|^{2(\tau-1)/\beta}}{\gamma^2 |\varepsilon|^{\tau-1}}, \quad \text{where } \beta := \frac{2 - \tau}{\tau}. \tag{88}$$

Proof

To obtain (88), we distinguish different cases.

First case: $|k-l| \geq (1/2)[\max(|k|,|l|)]^{\beta}$. *Then* $(\alpha_k\alpha_l)^{-1} \leq C|k-l|^{2(\tau-1)/\beta}/\gamma^2$. Indeed, we can estimate both α_k, α_l with the lower bound (84), $\alpha_k \geq c\gamma/|k|^{\tau-1}$, $\alpha_l \geq c\gamma/|l|^{\tau-1}$. Using the fact that $0<\beta<1$, we obtain

$$\frac{1}{\alpha_k\alpha_l} \leq C\frac{|k|^{\tau-1}|l|^{\tau-1}}{\gamma^2} \leq C\frac{[\max(|k|,|l|)]^{2(\tau-1)}}{\gamma^2} \leq C'\frac{|k-l|^{2(\tau-1)/\beta}}{\gamma^2}.$$

In the other cases, we have $0<|k-l|<(1/2)[\max(|k|,|l|)]^{\beta}$. We observe that in this situation, $\operatorname{sign}(l)=\operatorname{sign}(k)$, and to fix the ideas, we assume in the sequel that $l,k \geq 0$. (The estimate for $k,l<0$ is the same since $\alpha_k\alpha_l = \alpha_{-k}\alpha_{-l}$.) Moreover, since $\beta \leq 1$, we have $\max(k,l)=k$ or $l-k \leq (1/2)l^{\beta} \leq (1/2)l$. Hence $l \leq 2k$; similarly, $k \leq 2l$.

Second case: $0<|k-l|<(1/2)[\max(|k|,|l|)]^{\beta}$ *and* $(|k| \leq 1/3|\varepsilon|$ *or* $|l| \leq 1/3|\varepsilon|)$. *Then* $(\alpha_k\alpha_l)^{-1} \leq C/\gamma$. Suppose, for example, that $0 \leq k \leq 1/3|\varepsilon|$. We claim that if ε is small enough, then $\alpha_k \geq (k+1)/8$. Indeed, $\forall j \neq k$,

$$|\omega k - j| = |\omega k - k + k - j| \geq |k-j| - |\omega-1|\,|k| \geq 1 - 2|\varepsilon|\,k \geq \frac{1}{3}.$$

Therefore $\forall 1 \leq k < 1/3|\varepsilon|$, $\forall j \neq k$, $j \geq 1$, $|\omega^2k^2 - j^2| = |\omega k - j|\,|\omega k + j| \geq (\omega k+1)/3 \geq (k+1)/6$, and so

$$\begin{aligned}\alpha_k &:= \min_{j\geq 1, k\neq j}|\omega^2k^2 - \lambda_{k,j}| = \min_{j\geq 1,k\neq j}\left|\omega^2k^2 - j^2 - \varepsilon M(\delta,v_1,w) + O\Big(\frac{\varepsilon\|a_0\|_{H^1}}{j}\Big)\right| \\ &\geq \frac{k+1}{6} - |\varepsilon|\,C \geq \frac{k+1}{8}.\end{aligned}$$

Next, we estimate α_l. If $0 \leq l \leq 1/3|\varepsilon|$, then $\alpha_l \geq 1/8$ and therefore $(\alpha_k\alpha_l)^{-1} \leq 64$. If $l > 1/3|\varepsilon|$, we estimate α_l with the lower bound (84), and so, since $l \leq 2k$ and $1<\tau<2$,

$$\frac{1}{\alpha_k\alpha_l} \leq C\frac{l^{\tau-1}}{k\gamma} \leq \frac{C'}{k^{2-\tau}\gamma} \leq \frac{C'}{\gamma}.$$

In the remaining cases, we consider $|k-l|<(1/2)[\max(|k|,|l|)]^{\beta}$ and both $|k|,|l| > 1/3|\varepsilon|$. We have to distinguish two subcases. For this, $\forall k \in \mathbf{Z}$, let $j=j(k) \geq 1$ be an integer such that $\alpha_k := \min_{n\neq|k|}|\omega^2k^2 - \lambda_{k,n}| = |\omega^2k^2 - \lambda_{k,j}|$. Analogously, let $i=i(k) \geq 1$ be an integer such that $\alpha_l = |\omega^2l^2 - \lambda_{l,i}|$.

Third case: $0<|k-l|<(1/2)[\max(|k|,|l|)]^{\beta}$, $|k|,|l| > 1/3|\varepsilon|$, *and* $k-l=j-i$. *Then* $(\alpha_k\alpha_l)^{-1} \leq C/\gamma|\varepsilon|^{\tau-1}$. Indeed, $|(\omega k - j) - (\omega l - i)| = |\omega(k-l) - (j-i)| = |\omega-1||k-l| \geq |\varepsilon|/2$, and therefore $|\omega k - j| \geq |\varepsilon|/4$ or $|\omega l - i| \geq |\varepsilon|/4$. Assume, for instance, that $|\omega k - j| \geq |\varepsilon|/4$. Then $|\omega^2k^2 - j^2| = |\omega k - j|\,|\omega k + j| \geq |\varepsilon|\omega k/$

$2 \geq |\varepsilon|(1-2|\varepsilon|)k/2$, and so, for ε small enough, $|\alpha_k| \geq |\varepsilon|k/4$. Hence, since $l \leq 2k$ and $k > 1/3|\varepsilon|$,

$$\frac{1}{\alpha_k\alpha_l} \leq C\frac{l^{\tau-1}}{\gamma|\varepsilon|k} \leq \frac{C}{\gamma k^{2-\tau}|\varepsilon|} \leq \frac{C}{\gamma|\varepsilon|^{\tau-1}}.$$

Fourth case: $0 < |k-l| < (1/2)[\max(k,l)]^{\beta}$, $k, l > 1/3|\varepsilon|$, *and* $k-l \neq j-i$. *Then* $(\alpha_k\alpha_l)^{-1} \leq C/\gamma^2$. Using the fact that ω is γ-τ-Diophantine, we get

$$|(\omega k - j) - (\omega l - i)| = |\omega(k-l) - (j-i)| \geq \frac{\gamma}{|k-l|^{\tau}} \geq \frac{C\gamma}{[\max(k,l)]^{\beta\tau}} \geq \frac{C}{2}\Big(\frac{\gamma}{k^{\beta\tau}} + \frac{\gamma}{l^{\beta\tau}}\Big),$$

so that $|\omega k - j| \geq C\gamma/2k^{\beta\tau}$ or $|\omega l - i| \geq C\gamma/2l^{\beta\tau}$. Therefore $|\omega^2k^2 - j^2| \geq C'\gamma k^{1-\beta\tau} = C'\gamma k^{\tau-1}$ since $\beta := (2-\tau)/\tau$. Hence, for ε small enough, $\alpha_k \geq C'\gamma k^{\tau-1}/2$. We estimate α_l with the worst possible lower bound, and so, using also $l \leq 2k$, we obtain

$$\frac{1}{\alpha_k\alpha_l} \leq \frac{Cl^{\tau-1}}{\gamma^2 k^{\tau-1}} \leq \frac{C}{\gamma^2}.$$

Collecting the estimates of all the previous cases, (88) follows. □

Remark 4.1
The analysis of the “small divisors” in the second, third, and fourth cases of Lemma 4.5 corresponds, in the language of [11], to the property of *separation of the singular sites.*

LEMMA 4.6 (Bound of an off-diagonal operator)
Assume that $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$, *and let, for some* $s' \geq s$, $b(t,x) \in X_{\sigma,s'+(\tau-1)/\beta}$ *satisfy* $b_0(x) = 0$; *that is, let* $\int_0^{2\pi} b(t,x)\,dt \equiv 0$, $\forall x \in (0,\pi)$. *Define the operator* $T_n : W^{(n)} \to W^{(n)}$ *by*

$$T_n h := |D|^{-1/2} P_n \Pi_W\big(b(t,x)\,|D|^{-1/2}h\big).$$

There is a constant $\widetilde{C}$, *independent of* $b(t,x)$ *and of* n, *such that*

$$\|T_n h\|_{\sigma,s'} \leq \frac{\widetilde{C}}{|\varepsilon|^{(\tau-1)/2}\gamma}\|b\|_{\sigma,s'+(\tau-1)/\beta}\|h\|_{\sigma,s'}, \quad \forall h \in W^{(n)}.$$

Proof
For $h \in W^{(n)}$, we have $(T_n h)(t,x) = \sum_{|k|\leq L_n}(T_n h)_k(x)\exp(\mathrm{i}kt)$, with

$$(T_n h)_k = |D_k|^{-1/2}\pi_k(b\,|D|^{-1/2}h)_k = |D_k|^{-1/2}\pi_k\Big[\sum_{|l|\leq L_n} b_{k-l}|D_l|^{-1/2}h_l\Big]. \qquad (89)$$

Set $B_m := \|b_m(x)\|_{H^1}$. From (89) and (83), using the fact that $B_0 := \|b_0(x)\|_{H^1} = 0$,

$$\|(T_n h)_k\|_{H^1} \le C \sum_{|l|\le L_n, l\ne k} \frac{B_{k-l}}{\sqrt{\alpha_k}\sqrt{\alpha_l}} \|h_l\|_{H^1}. \tag{90}$$

Hence, by (88),

$$\|(T_n h)_k\|_{H^1} \le \frac{C}{\gamma|\varepsilon|^{(\tau-1)/2}} s_k, \quad \text{where } s_k := \sum_{|l|\le L_n} B_{k-l}|k-l|^{(\tau-1)/\beta}\|h_l\|_{H^1}. \tag{91}$$

By (91), setting $\widetilde{s}(t) := \sum_{|k|\le L_n} s_k \exp(ikt)$ (with $s_{-k} = s_k$),

$$\begin{aligned}\|T_n h\|^2_{\sigma,s'} &= \sum_{|k|\le L_n} \exp(2\sigma|k|)(k^{2s'}+1)\|(T_n h)_k\|^2_{H^1} \\ &\le \frac{C^2}{\gamma^2|\varepsilon|^{\tau-1}} \sum_{|k|\le L_n} \exp(2\sigma|k|)(k^{2s'}+1)s_k^2 = \frac{C^2}{\gamma^2|\varepsilon|^{\tau-1}}\|\widetilde{s}\|^2_{\sigma,s'}. \end{aligned} \tag{92}$$

It turns out that $\widetilde{s} = P_n(\widetilde{b}\widetilde{c})$, where $\widetilde{b}(t) := \sum_{l\in\mathbf{Z}} |l|^{(\tau-1)/\beta} B_l \exp(ilt)$ and $\widetilde{c}(t) := \sum_{|l|\le L_n} \|h_l\|_{H^1} \exp(ilt)$. Therefore, by (92) and since $s' > 1/2$,

$$\begin{aligned}\|T_n h\|_{\sigma,s'} &\le \frac{C}{\gamma|\varepsilon|^{(\tau-1)/2}}\|\widetilde{b}\widetilde{c}\|_{\sigma,s'} \le \frac{C}{\gamma|\varepsilon|^{(\tau-1)/2}}\|\widetilde{b}\|_{\sigma,s'}\|\widetilde{c}\|_{\sigma,s'} \\ &\le \frac{C}{\gamma|\varepsilon|^{(\tau-1)/2}}\|b\|_{\sigma,s'+(\tau-1)/\beta}\|h\|_{\sigma,s'}\end{aligned}$$

since $\|\widetilde{b}\|_{\sigma,s'} \le \|b\|_{\sigma,s'+(\tau-1)/\beta}$ and $\|\widetilde{c}\|_{\sigma,s'} = \|h\|_{\sigma,s'}$. □

Before proving the smallness of the *off-diagonal* operator $\mathscr{R}_1$ and of $\mathscr{R}_2$, we need the following preliminary lemma, which gives a suitable estimate of the multiplicative function $a(t,x)$.

LEMMA 4.7
There are $\mu > 0$, $\delta_0 > 0$, and $C > 0$ with the following property: if $\|v_1\|_{0,s} \le 2R$, $[w]_{\sigma,s} \le \mu$, and $\delta \in [0,\delta_0)$, then $\|a\|_{\sigma,s+2(\tau-1)/\beta} \le C$.

Proof
By Definition 3.2 of $[w]_{\sigma,s}$, there are $h_i \in W^{(i)}$, $0 \le i \le q$, and a sequence $(\sigma_i)_{0\le i\le q}$ with $\sigma_i > \sigma$ such that $w = h_0 + h_1 + \cdots + h_q$ and

$$\sum_{i=0}^{q} \frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i-\sigma)^{2(\tau-1)/\beta}} \le 2[w]_{\sigma,s} \le 2\mu. \tag{93}$$

An elementary calculus, using the fact that $\max_{k\geq 1} k^{\alpha} \exp\{-(\sigma_i - \sigma)k\} \leq C(\alpha)/(\sigma_i - \sigma)^{\alpha}$, gives

$$\|h_i\|_{\sigma,s+2(\tau-1)/\beta} \leq C(\tau)\frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i-\sigma)^{2(\tau-1)/\beta}}. \tag{94}$$

Hence, by (93) and (94),

$$\|w\|_{\sigma,s+2(\tau-1)/\beta} \leq \sum_{i=0}^{q} \|h_i\|_{\sigma,s+2(\tau-1)/\beta} \leq \sum_{i=0}^{q} C(\tau)\frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i-\sigma)^{2(\tau-1)/\beta}} \leq C(\tau)2\mu.$$

By Lemma 2.1(d), provided δ_0 is small enough, also $\|v_2(\delta, v_1, w)\|_{\sigma,s+2(\tau-1)/\beta} \leq C'$, and therefore

$$\|a\|_{\sigma,s+2(\tau-1)/\beta} = \left\|(\partial_u g)\big(\delta, x, v_1 + w + v_2(\delta, v_1, w)\big)\right\|_{\sigma,s+2(\tau-1)/\beta} \leq C.$$

This bound is a consequence of the analyticity assumption **(H)** on the nonlinearity f, the Banach algebra property of $X_{\sigma,s+2(\tau-1)/\beta}$, and can be obtained as in (22). □

LEMMA 4.8 (Estimate of $\mathscr{R}_1$)
Under the hypotheses of (P3), there exists a constant $C > 0$ depending on μ such that

$$\|\mathscr{R}_1 h\|_{\sigma,s+(\tau-1)/2} \leq |\varepsilon|^{(3-\tau)/2}\frac{C}{\gamma}\|h\|_{\sigma,s+(\tau-1)/2}, \quad \forall h \in W^{(n)}.$$

Proof
Recalling the definition of $\mathscr{R}_1 := |D|^{-1/2}\mathscr{M}_1|D|^{-1/2}$ and $\mathscr{M}_1$, and using Lemma 4.6 since $\overline{a}(t, x)$ has zero time-average,

$$\begin{aligned}
&\|\mathscr{R}_1 h\|_{\sigma,s+(\tau-1)/2}\\
&\quad = \|\,|D|^{-1/2}\mathscr{M}_1|D|^{-1/2}h\|_{\sigma,s+(\tau-1)/2} = |\varepsilon|\,\left\||D|^{-1/2}P_n\Pi_W(\overline{a}\,|D|^{-1/2}h)\right\|_{\sigma,s+(\tau-1)/2}\\
&\quad \leq |\varepsilon|\,\frac{\widetilde{C}}{|\varepsilon|^{(\tau-1)/2}\gamma}\|\overline{a}\|_{\sigma,s+(\tau-1)/2+(\tau-1)/\beta}\|h\|_{\sigma,s+(\tau-1)/2} \leq |\varepsilon|^{(3-\tau)/2}\,\frac{\widetilde{C}}{\gamma}\|\overline{a}\|_{\sigma,s+2(\tau-1)/\beta}\|h\|_{\sigma,s+(\tau-1)/2}\\
&\quad \leq |\varepsilon|^{(3-\tau)/2}\frac{C}{\gamma}\|h\|_{\sigma,s+(\tau-1)/2}
\end{aligned}$$

since $0 < \beta < 1$ and, by Lemma 4.7, $\|\overline{a}\|_{\sigma,s+2(\tau-1)/\beta} \leq \|a\|_{\sigma,s+2(\tau-1)/\beta} \leq C$. □

The *smallness* of $\mathscr{R}_2 := |D|^{-1/2}\mathscr{M}_2|D|^{-1/2}$ with respect to U is just a consequence of Lemma 4.7 and of the regularizing property of $\partial_w v_2 : X_{\sigma,s} \to X_{\sigma,s+2}$ proved in Lemma 2.1. By (86), the *loss of* $\tau - 1$ *derivatives* due to $|D|^{-1/2}$ applied twice is compensated by the gain of two derivatives due to $\partial_w v_2 : X_{\sigma,s} \to X_{\sigma,s+2}$.

LEMMA 4.9 (Estimate of $\mathscr{R}_2$)
Under the hypotheses of (P3), there exists a constant $C > 0$ depending on μ such that

$$\|\mathscr{R}_2 h\|_{\sigma,s+(\tau-1)/2} \le C\frac{|\varepsilon|}{\gamma}\,\|h\|_{\sigma,s+(\tau-1)/2}, \quad \forall h \in W^{(n)}.$$

Proof
By (86) and the regularizing estimate $\|\partial_w v_2[u]\|_{\sigma,s+2} \le C\|u\|_{\sigma,s}$ of Lemma 2.1, we get

$$\begin{aligned}
\|\mathscr{R}_2 h\|_{\sigma,s+(\tau-1)/2} &\le \frac{C}{\sqrt{\gamma}}\|\mathscr{M}_2|D|^{-1/2}h\|_{\sigma,s+\tau-1} \\
&= C\frac{|\varepsilon|}{\sqrt{\gamma}}\,\Big\|P_n\Pi_W\big(a\ \partial_w v_2[|D|^{-1/2}h]\big)\Big\|_{\sigma,s+\tau-1} \\
&\le C\frac{|\varepsilon|}{\sqrt{\gamma}}\,\|a\|_{\sigma,s+\tau-1}\Big\|\partial_w v_2[|D|^{-1/2}h]\Big\|_{\sigma,s+\tau-1} \\
&\le C'\frac{|\varepsilon|}{\sqrt{\gamma}}\,\|a\|_{\sigma,s+\tau-1}\Big\|\partial_w v_2[|D|^{-1/2}h]\Big\|_{\sigma,s+2} \\
&\le C\frac{|\varepsilon|}{\sqrt{\gamma}}\,\|a\|_{\sigma,s+\tau-1}\||D|^{-1/2}h\|_{\sigma,s} \le C'\frac{|\varepsilon|}{\gamma}\,\|h\|_{\sigma,s+(\tau-1)/2}
\end{aligned}$$

since $\tau < 3$ and, by Lemma 4.7, $\|a\|_{\sigma,s+\tau-1} \le \|a\|_{\sigma,s+2(\tau-1)/\beta} \le C$. □

Proof of property (P3) completed
Under the hypothesis of (P3), the linear operator U is invertible by Lemma 4.4 and, by Lemmas 4.9 and 4.8, provided that δ is small enough,

$$\|U^{-1}\mathscr{R}_1\|_{\sigma,s+(\tau-1)/2},\ \|U^{-1}\mathscr{R}_2\|_{\sigma,s+(\tau-1)/2} < \frac{1}{4}.$$

Therefore also the linear operator $U - \mathscr{R}_1 - \mathscr{R}_2$ is invertible, and its inverse satisfies

$$\|(U-\mathscr{R}_1-\mathscr{R}_2)^{-1}h\|_{\sigma,s+(\tau-1)/2} = \|(I-U^{-1}\mathscr{R}_1-U^{-1}\mathscr{R}_2)^{-1}U^{-1}h\|_{\sigma,s+(\tau-1)/2} \tag{95}$$

$$\le 2\|U^{-1}h\|_{\sigma,s+(\tau-1)/2} \le C\|h\|_{\sigma,s+(\tau-1)/2}, \quad \forall h \in W^{(n)}. \tag{96}$$

Hence $\mathscr{L}_n$ is invertible, $\mathscr{L}_n^{-1} = |D|^{-1/2}(U-\mathscr{R}_1-\mathscr{R}_2)^{-1}|D|^{-1/2} : W^{(n)} \to W^{(n)}$, and by (86), (95),

$$\begin{aligned}
\|\mathscr{L}_n^{-1}h\|_{\sigma,s} &= \||D|^{-1/2}(U-\mathscr{R}_1-\mathscr{R}_2)^{-1}|D|^{-1/2}h\|_{\sigma,s} \\
&\le \frac{C}{\sqrt{\gamma}}\|(U-\mathscr{R}_1-\mathscr{R}_2)^{-1}|D|^{-1/2}h\|_{\sigma,s+(\tau-1)/2} \\
&\le \frac{C'}{\sqrt{\gamma}}\||D|^{-1/2}h\|_{\sigma,s+(\tau-1)/2} \le \frac{C''}{\gamma}\|h\|_{\sigma,s+\tau-1} \le \frac{C''}{\gamma}(L_n)^{\tau-1}\|h\|_{\sigma,s}
\end{aligned}$$

because $h \in W^{(n)}$. This completes the proof of property (P3). □

5. Solution of the $(Q1)$-equation

Once the $(Q2)$ and (P)-equations are solved (with *gaps* for the latter), the last step is to find solutions of the finite-dimensional $(Q1)$-equation

$$- \Delta v_1 = \Pi_{V_1} \mathscr{G}(\delta, v_1), \tag{97}$$

where

$$\mathscr{G}(\delta, v_1)(t, x) := g\big(\delta, x, v_1(t, x) + \widetilde{w}(\delta, v_1)(t, x) + v_2(\delta, v_1, \widetilde{w}(\delta, v_1))(t, x)\big).$$

We are interested in solutions (δ, v_1) which belong to the Cantor set B_∞.

5.1. The $(Q1)$-equation for $\delta = 0$

For $\delta = 0$, the $(Q1)$-equation (97) reduces to

$$- \Delta v_1 = \Pi_{V_1} \mathscr{G}(0, v_1) = s^* \Pi_{V_1} \big(a_p(x)(v_1 + v_2(0, v_1, 0))^p\big), \tag{98}$$

which is the Euler-Lagrange equation of $\Psi_0 : B(2R, V_1) \to \mathbf{R}$,

$$\Psi_0(v_1) := \Phi_0\big(v_1 + v_2(0, v_1, 0)\big), \tag{99}$$

where $\Phi_0 : V \to \mathbf{R}$ is defined in (12).

In fact, since $v_2(0, v_1, 0)$ solves the $(Q2)$-equation (for $\delta = 0$, $w = 0$), $d\Phi_0(v_1 + v_2(0, v_1, 0))[k] = 0$, $\forall k \in V_2$. Moreover, since $\forall h \in V_1$, $D_{v_1} v_2(0, v_1, 0)[h] \in V_2$,

$$\begin{aligned} d\Psi_0(v_1)[h] &= d\Phi_0\big(v_1 + v_2(0, v_1, 0)\big)\big[h + D_{v_1} v_2(0, v_1, 0)[h]\big] \\ &= d\Phi_0\big(v_1 + v_2(0, v_1, 0)\big)[h] \\ &= \int_\Omega \big[- \Delta v_1 - s^* \Pi_{V_1} \big(a_p(x)(v_1 + v_2(0, v_1, 0))\big)^p \big] h. \end{aligned} \tag{100}$$

Hence $\overline{v}_1$ is a critical point of Ψ_0 if and only if it is a solution of equation (98).

LEMMA 5.1

Let $\overline{v}$ be the nondegenerate solution of equation (11) introduced in Theorem 1.2. Then $\overline{v}_1 := \Pi_{V_1} \overline{v} \in B(R; V_1)$ is a nondegenerate solution of (98).

Proof

By Lemma 2.1(b), $\Pi_{V_2} \overline{v} = v_2(0, \overline{v}_1, 0)$. Hence, since $\overline{v}$ solves (11), $\overline{v}_1$ solves (98). Now assume that $h_1 \in V_1$ is a solution of the linearized equation at $\overline{v}_1$ of (98). This means

$$- \Delta h_1 = s^* \Pi_{V_1} \big(p a_p(x)(\overline{v}_1 + v_2(0, \overline{v}_1, 0))^{p-1}(h_1 + h_2)\big), \tag{101}$$

where $h_2 := D_{v_1} v_2(0, \overline{v}_1, 0)[h_1] \in V_2$. Now, by the definition of the map v_2, we have

$$-\Delta v_2(0, v_1, 0) = s^* \Pi_{V_2}\big(a_p(x)(v_1 + v_2(0, v_1, 0))^p\big), \quad \forall v_1 \in B(2R, V_1),$$

from which we derive, taking the differential at $\overline{v}_1$,

$$-\Delta h_2 = s^* \Pi_{V_2}\big(p a_p(x)(\overline{v}_1 + v_2(0, \overline{v}_1, 0))^{p-1}(h_1 + h_2)\big). \tag{102}$$

Summing (101) and (102), we obtain that $h = h_1 + h_2$ is a solution of the linearized form at $\overline{v}$ of equation (11). Since $\overline{v}$ is a nondegenerate solution of (11), $h = 0$; hence $h_1 = 0$. As a result, $\overline{v}_1 = \Pi_{V_1}\overline{v}$ is a nondegenerate solution of (98). □

5.2. Proof of Theorem 1.2

By assumption, $\overline{v}$ is a nondegenerate solution of equation (11). Hence, by Lemma 5.1, $\overline{v}_1 = \Pi_{V_1}\overline{v} \in B(R, V_1)$ is a nondegenerate solution of (98).

Since the map $(\delta, v_1) \to -\Delta v_1 - \Pi_{V_1}\mathscr{G}(\delta, v_1)$ is in $C^\infty([0, \delta_0) \times V_1; V_1)$, by the implicit function theorem there is a C^∞-path

$$\delta \mapsto v_1(\delta) \in B(2R, V_1)$$

such that $v_1(\delta)$ is a solution of (97) and $v_1(0) = \overline{v}_1$.

By Theorem 3.1, the function

$$\widetilde{u}(\delta) := \delta\big[v_1(\delta) + v_2\big(\delta, v_1(\delta), \widetilde{w}(\delta, v_1(\delta))\big) + \widetilde{w}\big(\delta, v_1(\delta)\big)\big] \in X_{\overline{\sigma}/2,s} \tag{103}$$

is a solution of equation (3) if δ belongs to the Cantor-like set

$$\mathscr{C} := \Big\{\delta \in [0, \delta_0) \,\Big|\, \big(\delta, v_1(\delta)\big) \in B_\infty\Big\}.$$

By Proposition 3.2, the smoothness of $v_1(\cdot)$ implies that the Cantor set $\mathscr{C}$ has full density at the origin (i.e., satisfies the measure estimate (4)).

Finally, by (103), since $\overline{v} = \overline{v}_1 + v_2(0, \overline{v}_1, 0)$,

$$\begin{aligned}
&\|\widetilde{u}(\delta) - \delta\overline{v}\|_{\overline{\sigma}/2,s} \\
&\quad = \delta\Big\|\big(v_1(\delta) - \overline{v}_1\big) + \big(v_2(\delta, v_1(\delta), \widetilde{w}(\delta, v_1(\delta))) - v_2(0, \overline{v}_1, 0)\big) + \widetilde{w}\big(\delta, v_1(\delta)\big)\Big\|_{\overline{\sigma}/2,s} \\
&\quad \le \delta\Big(\|v_1(\delta) - \overline{v}_1\|_{\overline{\sigma}/2,s} + \Big\|v_2(\delta, v_1(\delta), \widetilde{w}(\delta, v_1(\delta))) - v_2(0, \overline{v}_1, 0)\Big\|_{\overline{\sigma}/2,s} \\
&\qquad\quad + \|\widetilde{w}(\delta, v_1(\delta))\|_{\overline{\sigma}/2,s}\Big) = O(\delta^2)
\end{aligned}$$

by (57).

This proves Theorem 1.2 in the case when $\overline{v}$ is nondegenerate in the whole space V.

Now, we can look for $(2\pi/n)$-time-periodic solutions of (3) as well. (They are particular 2π-periodic solutions.) Let

$$X_{\sigma,s,n} := \left\{ u \in X_{\sigma,s} \,\middle|\, u \text{ is } \frac{2\pi}{n} \text{ time-periodic} \right\} = V_n \oplus W_n,$$

where V_n (defined in (14)) and W_n are the subspaces of V and W formed by the functions $(2\pi/n)$-periodic in t.

Introducing an appropriate finite-dimensional subspace $V_{1,n} \subset V_n$, we split $V_n = V_{1,2} \oplus V_{2,n}$, and we obtain associated $(Q1)$-, $(Q2)$-, (P)-equations as in (15).

With the arguments of Sections 2 and 3, we can solve the $(Q2)$- and (P)-equations exactly as in the case where $n = 1$.

The zeroth-order bifurcation equation is again equation (11) but in V_n, and the corresponding functional is just the restriction of Φ_0 to V_n.

The main assumption of Theorem 1.2 (that at least one of the critical points of $(\Phi_0)_{|V_n}$, called $\overline{v}$, is nondegenerate) allows us to find a C^∞-path $\delta \mapsto v_1(\delta) \in V_{1,n}$ of solutions of equation (97).

As above, this implies the conclusions of Theorem 1.2. □

6. Proof of Theorem 1.1

For this section we define the linear map $\mathscr{H}_n : V \to V$ by

$$\text{for } v(t,x) = \eta(t+x) - \eta(t-x) \in V, \quad (\mathscr{H}_n v)(t,x) := \eta\big(n(t+x)\big) - \eta\big(n(t-x)\big)$$

so that $V_n = \mathscr{H}_n V$.

6.1. Case $f(x,u) = a_3(x)u^3 + O(u^4)$

LEMMA 6.1

Let $\langle a_3 \rangle := (1/\pi)\int_0^\pi a_3(x) \neq 0$. Taking $s^* = \operatorname{sign}\langle a_3 \rangle$, $\exists\, n_0 \in \mathbf{N}$ such that $\forall n \geq n_0$, the zeroth-order bifurcation equation (16) has a solution $\overline{v} \in V_n$ which is nondegenerate in V_n.

Proof

Equation (16) is the Euler-Lagrange equation of

$$\Phi_0(v) = \frac{\|v\|_{H^1}^2}{2} - s^* \int_\Omega a_3(x) \frac{v^4}{4}. \tag{104}$$

The functional $\Phi_n(v) := \Phi_0(\mathscr{H}_n v)$ has the following development: for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$ we obtain, using the fact that $\int_\Omega v^4 = \int_\Omega (\mathscr{H}_n v)^4$,

$$\Phi_n(v) = 2\pi n^2 \int_{\mathbf{T}} \dot{\eta}^2(t)\,dt - s^* \langle a_3 \rangle \int_\Omega \frac{v^4}{4} - s^* \int_\Omega \big(a_3(x) - \langle a_3 \rangle\big) \frac{(\mathscr{H}_n v)^4}{4}.$$

We choose $s^* = \operatorname{sign}\langle a_3\rangle$ so that $s^*\langle a_3\rangle > 0$. To simplify notation, take $\langle a_3\rangle > 0$ so that $s^* = 1$,

$$\Phi_n\Big(\frac{\sqrt{2}n}{\sqrt{\langle a_3\rangle}}v\Big) = \frac{8\pi n^4}{\langle a_3\rangle}\Big[\frac{1}{2}\int_{\mathbf{T}}\dot\eta^2(s)\,ds - \frac{1}{8\pi}\int_\Omega v^4 + \frac{1}{8\pi}\int_\Omega\Big(\frac{a_3(x)}{\langle a_3\rangle}-1\Big)(\mathscr{H}_n v)^4\,dt\,dx\Big]$$
$$= \frac{8\pi n^4}{\langle a_3\rangle}[\Psi(\eta)+\mathscr{R}_n(v)],$$

where

$$\Psi(\eta) := \frac{1}{2}\int_{\mathbf{T}}\dot\eta^2(s)\,ds - \frac{1}{4}\int_{\mathbf{T}}\eta^4(s)\,ds - \frac{3}{8\pi}\Big(\int_{\mathbf{T}}\eta^2(s)\,ds\Big)^2,$$

$$\mathscr{R}_n(v) := \frac{1}{8\pi}\int_\Omega b(x)(\mathscr{H}_n v)^4\,dt\,dx, \quad b(x) := \frac{a_3(x)}{\langle a_3\rangle} - 1.$$

Let $E := \{\eta \in H^1(\mathbf{T}) \mid \eta \text{ is odd}\}$. It is enough to prove that $\Psi : E \to \mathbf{R}$ has a nondegenerate critical point $\overline{\eta}$ and that $\mathscr{R}_n$ is small for large n (see Lemma 6.2). Indeed, the operator $\Psi''(\overline{\eta})$ has the form Id + Compact, so that if its kernel is zero, then $\Psi''(\overline{\eta})$ is invertible. Hence, by the implicit function theorem, for n large enough, Φ_n too (hence $\Phi_{0|V_n}$) has a nondegenerate critical point.

The critical points of Ψ in E are the 2π-periodic odd solutions of

$$\ddot\eta + \eta^3 + 3\langle\eta^2\rangle\eta = 0. \tag{105}$$

By [2] it is known that there exists a solution of (105) which is a nondegenerate critical point of Ψ in E. It remains to prove Lemma 6.2. □

LEMMA 6.2
There holds

$$\|D\mathscr{R}_n(v)\|,\ \|D^2\mathscr{R}_n(v)\| \to 0 \quad \text{as } n \to +\infty \tag{106}$$

uniformly for v in bounded sets of E.

Proof
We prove the estimate only for $D^2\mathscr{R}_n$. We have

$$D^2\mathscr{R}_n(v)[h,k] = \frac{3}{2\pi}\int_\Omega b(x)(\mathscr{H}_n v)^2(\mathscr{H}_n h)\,(\mathscr{H}_n k) = \frac{3}{2\pi}\int_0^\pi b(x)g(nx)\,dx,$$

where $g(y)$ is the π-periodic function defined by

$$g(y) := \int_{\mathbf{T}} \big(\eta(t+y) - \eta(t-y)\big)^2 \big(\beta(t+y) - \beta(t-y)\big)\big(\gamma(t+y) - \gamma(t-y)\big)\, dt,$$

β and γ being associated with h and k as η is with v. Developing in Fourier series $g(y) = \sum_{l\in\mathbf{Z}} g_l \exp(\mathrm{i}2ly)$, we have $g(nx) = \sum_{l\in\mathbf{Z}} g_l \exp(\mathrm{i}2lnx)$. Extending $b(x)$ to a π-periodic function, we also write $b(x) = \sum_{l\in\mathbf{Z}} b_l \exp(\mathrm{i}2lx)$ with $b_0 = \langle b\rangle = 0$. Therefore

$$\begin{aligned}
&|D^2\mathcal{R}_n(v_n)[h,k]| \\
&\quad = \frac{3}{2}\Big|\sum_{l\neq 0} g_l b_{-ln}\Big| \leq \frac{3}{2}\Big(\sum_{l\neq 0} g_l^2\Big)^{1/2}\Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2} \leq \frac{3}{2}\|g\|_{L^2(0,\pi)}\Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2} \\
&\quad \leq C\|\eta\|_\infty^2 \|\beta\|_\infty \|\gamma\|_\infty \Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2} \leq C\|v_0\|_{H^1}^2 \|h\|_{H^1}\|k\|_{H^1} \Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2}.
\end{aligned}$$

Since $\big(\sum_{l\neq 0} b_{ln}^2\big)^{1/2} \to 0$ as $n \to \infty$, it proves (106). With a similar calculus, we can prove that $D\mathcal{R}_n(v) \to 0$ as $n \to +\infty$. □

6.2. Case $f(x,u) = a_2u^2 + O(u^4)$

With the frequency-amplitude relation (17), system (7) with $p = 2$ becomes

$$\begin{cases} -\Delta v = -\delta^{-1}\Pi_V g_\delta(x, v+w), & (Q) \\ L_\omega w = \delta \Pi_W g_\delta(x, v+w), & (P) \end{cases} \tag{107}$$

where

$$g_\delta(x,u) = \frac{f(x,\delta u)}{\delta^2} = a_2u^2 + \delta^2 a_4(x)u^4 + \cdots. \tag{108}$$

With the further rescaling

$$w \to \delta w$$

and since $v^2 \in W$, system (107) is equivalent to

$$\begin{cases} -\Delta v = \Pi_V\big(-2a_2vw - a_2\delta w^2 - \delta r(\delta, x, v+\delta w)\big), & (Q) \\ L_\omega w = a_2v^2 + \delta\Pi_W\big(2a_2vw + \delta a_2 w^2 + \delta r(\delta, x, v+\delta w)\big), & (P) \end{cases} \tag{109}$$

where $r(\delta, x, u) = \delta^{-4}(f(x,\delta u) - a_2\delta^2u^2) = a_4(x)u^4 + \cdots$.

For $\delta = 0$, system (109) reduces to

$$\begin{cases} -\Delta v = -2a_2\Pi_V(vw), \\ Lw = a_2v^2, \end{cases} \tag{110}$$

where $L := -\partial_{tt} + \partial_{xx}$, and it is equivalent to $w = a_2 L^{-1} v^2$, $-\Delta v = -2a_2^2 \Pi_V (vL^{-1}v^2)$, namely, to the zeroth-order bifurcation equation (18).

LEMMA 6.3
If $a_2 \neq 0$, $\exists\, n_0 \in \mathbf{N}$ such that $\forall n \geq n_0$, the zeroth-order bifurcation equation (18) has a solution $\overline{v} \in V_n$ which is nondegenerate in V_n.

Proof
We have to prove that $\Phi_n(v) := \Phi_0(\mathscr{H}_n v)$, where Φ_0 is defined in (19), possesses nondegenerate critical points at least for n large.

Φ_n admits the following development (see [5, Lemmas 3.7, 3.8]). For $v(t,x) = \eta(t+x) - \eta(t-x)$,

$$\Phi_n(v) = 2\pi n^2 \int_{\mathbf{T}} \dot{\eta}^2(t)\, dt - \frac{\pi^2 a_2^2}{12}\Big(\int_{\mathbf{T}} \eta^2(t)\, dt\Big)^2 + \frac{a_2^2}{2n^2}\Big(\int_{\Omega} v^2 L^{-1} v^2 + \frac{\pi^2}{6}\Big(\int_{\mathbf{T}} \eta^2(t)\, dt\Big)^2\Big).$$

Hence we can write

$$\begin{aligned}\Phi_n\Big(\frac{\sqrt{12}n}{\sqrt{\pi}a_2} v\Big) &= \frac{48n^4}{a_2^2}\Big[\frac{1}{2}\int_{\mathbf{T}} \dot{\eta}^2(s)\, ds - \frac{1}{4}\Big(\int_{\mathbf{T}} \eta^2(s)\, ds\Big)^2 + \frac{1}{n^2}\mathscr{R}(\eta)\Big] \\ &= \frac{48n^4}{a_2^2}\Big[\Psi(\eta) + \frac{1}{n^2}\mathscr{R}(\eta)\Big], \end{aligned} \tag{111}$$

where

$$\Psi(\eta) = \frac{1}{2}\int_{\mathbf{T}} \dot{\eta}^2(s)\, ds - \frac{1}{4}\Big(\int_{\mathbf{T}} \eta^2(s)\, ds\Big)^2$$

and $\mathscr{R} : E \to \mathbf{R}$ is a smooth functional defined on $E := \{\eta \in H^1(\mathbf{T}) \mid \eta \text{ odd}\}$. By (111), in order to prove that Φ_n has a nondegenerate critical point for n large enough, it is enough to prove the following lemma. □

LEMMA 6.4
$\Psi : E \to \mathbf{R}$ possesses a nondegenerate critical point.

Proof
The critical points of Ψ in E are the 2π-periodic odd solutions of the equation

$$\ddot{\eta} + \Big(\int_{\mathbf{T}} \eta^2(t)\, dt\Big)\eta = 0. \tag{112}$$

Equation (112) has a 2π-periodic solution of the form $\bar{\eta}(t) = (1/\sqrt{\pi})\sin t$.

We claim that $\bar{\eta}$ is nondegenerate. The linearized equation of (112) at $\bar{\eta}$ is

$$\ddot{h} + h + \frac{2}{\pi}\Big(\int_{\mathbf{T}} \sin t\, h(t)\, dt\Big) \sin t = 0. \tag{113}$$

Developing in time-Fourier series $h(t) = \sum_{k\geq 1} a_k \sin kt$, we find out that any solution of the linearized equation (113) satisfies

$$-k^2 a_k + a_k = 0, \quad \forall k \geq 2, \qquad a_1 = 0$$

and therefore $h = 0$. □

As in Theorem 1.2, the existence of a solution $\overline{v}$ of the zeroth-order bifurcation equation which is nondegenerate in some V_n entails the conclusions of Theorem 1.2. To avoid cumbersome notation, we still give the main arguments assuming that $n = 1$.

Since for $\delta = 0$ the solution of the (P)-equation in (110) is $w = a_2 L^{-1} v^2$, it is convenient to perform the change of variable

$$w = a_2 L^{-1} v^2 + y, \quad y \in W. \tag{114}$$

System (109) is then written

$$\begin{cases} -\Delta v = -2a_2^2 \Pi_V (v L^{-1} v^2) + \Pi_V\big(-2a_2 v y - a_2 \delta w^2 - \delta r(\delta, x, v + \delta w)\big), & (Q') \\ L_\omega y = 2a_2 \delta^2 \mathscr{R}(v^2) + \delta \Pi_W\big(2a_2 v w + \delta a_2 w^2 + \delta r(\delta, x, v + \delta w)\big), & (P') \end{cases} \tag{115}$$

where w is a function of v and y through (114), and the linear operator in W,

$$\mathscr{R} := (1 - \omega^2)^{-1}(I - L_\omega L^{-1}) = (2\delta^2)^{-1}(I - L_\omega L^{-1}),$$

does not depend on ω and can be expressed as

$$\mathscr{R}\Big(\sum_{l\neq j} w_{l,j} \cos(lt)\sin(jx)\Big) = \sum_{l\neq j} \frac{l^2}{l^2 - j^2} w_{l,j} \cos(lt)\sin(jx).$$

Since $l^2|l^2 - j^2|^{-1} = l^2|l + j|^{-1}|l - j|^{-1} \leq |l|$, the operator $\mathscr{R}$ satisfies the estimate

$$\forall w \in W, \quad \|\mathscr{R} w\|_{\sigma,s} \leq \|w\|_{\sigma,s+1}. \tag{116}$$

Splitting $V = V_1 \oplus V_2$, the (Q')-equation is divided in two parts: the $(Q'1)$- and $(Q'2)$-equations.

Setting

$$R := \|\overline{v}\|_{0,s},$$

the analogue of Lemma 2.1 is the following.

LEMMA 6.5
There exist $N \in \mathbf{N}_+$, $\overline{\sigma} = \ln 2/N > 0$, $\delta_0 > 0$, *such that*, $\forall\, 0 \le \sigma \le \overline{\sigma}$, $\forall\, \|v_1\|_{0,s} \le 2R$, $\forall\, \|y\|_{\sigma,s} \le 1$, $\forall\, \delta \in [0, \delta_0)$, *there exists a unique solution* $v_2(\delta, v_1, y) \in V_2 \cap X_{\sigma,s}$ *of the* $(Q'2)$*-equation with* $\|v_2(\delta, v_1, y)\|_{\sigma,s} \le 1$. *Moreover,* $v_2(0, \Pi_{V_1}\overline{v}, 0) = \Pi_{V_2}\overline{v}$, $v_2(\delta, v_1, y) \in X_{\sigma,s+2}$, *and the regularizing property*

$$\|D_w v_2(\delta, v_1, y)[h]\|_{\sigma,s+2} \le C\|h\|_{\sigma,s} \tag{117}$$

holds, where C *is some positive constant.*

Substituting $v_2 = v_2(\delta, v_1, y)$ into the (P')-equation yields

$$L_\omega y = \delta\Gamma(\delta, v_1, y) := \delta\widetilde{\Gamma}\big(\delta, v_1 + v_2(\delta, v_1, y), y\big), \tag{118}$$

where

$$\widetilde{\Gamma}(\delta, v, y) := 2\delta a_2\mathscr{R}(v^2) + \Pi_W\big(2a_2 v(a_2 L^{-1}(v^2) + y) + \delta a_2(a_2 L^{-1}(v^2) + y)^2 + \delta r(\delta, x, v + \delta(a_2 L^{-1}(v^2) + y))\big).$$

The (P')-equation (118) can be solved as in Sections 3 and 4 with slight changes that we specify.

THEOREM 6.1 (Solution of the (P')-equation)
For $\delta_0 > 0$ *small enough, there exists a* C^∞*-function* $\widetilde{y} : [0, \delta_0) \times B(2R, V_1) \to W \cap X_{\overline{\sigma}/2,s}$ *satisfying* $\widetilde{y}(0, v_1) = 0$, $\|\widetilde{y}\|_{\overline{\sigma}/2,s} = O(\delta)$, $\|D^k\widetilde{y}\|_{\overline{\sigma}/2,s} = O(1)$, *and verifying the following property. Let*

$$B_\infty := \Big\{(\delta, v_1) \in [0, \delta_0) \times B(2R, V_1) : \Big|\omega(\delta)l - j - \delta\frac{M(\delta, v_1, \widetilde{y}(\delta, v_1))}{2j}\Big| \ge \frac{2\gamma}{(l+j)^\tau}, \quad |\omega(\delta)l - j| \ge \frac{2\gamma}{(l+j)^\tau}, \forall l \ge \frac{1}{3\delta^2},\ l \ne j\Big\},$$

where $\omega(\delta) = \sqrt{1 - 2\delta^2}$ *and* $M(\delta, v_1, y)$ *is defined in (119). Then* $\forall(\delta, v_1) \in B_\infty$, $\widetilde{y}(\delta, v_1)$ *solves the* (P')*-equation (118).*

Proof
As before, the key point is the inversion, at each step of the iterative process, of a linear operator

$$\mathscr{L}_n(\delta, v_1, y)[h] = L_\omega h - \delta P_n \Pi_W D_y\Gamma(\delta, v_1, y)[h], \quad h \in W^{(n)}.$$

We have

$$D_y\Gamma(\delta, v_1, y)[h]$$
$$= D_y\widetilde{\Gamma}\big(\delta, v_1 + v_2(\delta, v_1, y), y\big)[h] + D_v\widetilde{\Gamma}\big(\delta, v_1 + v_2(\delta, v_1, y), y\big)D_y v_2(\delta, v_1, y)[h]$$

and, as it can be directly verified,

$$D_y\widetilde{\Gamma}(\delta, v, y)[h] = \Pi_W\big((\partial_u g_\delta)(x, v + \delta w)h\big),$$

where g_δ is defined in (108) and w is given by (114). As in Section 4, setting $a(t, x) := (\partial_u g_\delta)(x, v(t, x) + \delta w(t, x))$, we can decompose $\mathscr{L}_n(\delta, v_1, y) = D - \mathscr{M}_1 - \mathscr{M}_2$, where (with the notation of Section 4)

$$\begin{cases} Dh := L_\omega h - \delta P_n \Pi_W\big(a_0(x)h\big), \\ \mathscr{M}_1 h := \delta P_n \Pi_W\big(\overline{a}(t, x)h\big), \\ \mathscr{M}_2 h := \delta P_n \Pi_W D_v \widetilde{\Gamma}\big(\delta, v_1 + v_2(\delta, v_1, y), y\big)D_y v_2(\delta, v_1, y)[h]. \end{cases}$$

As in Lemma 4.1, the eigenvalues of the similarly defined operator S_k satisfy $\lambda_{k,j} = j^2 + \delta M(\delta, v_1, y) + O(\delta/j)$, where

$$M(\delta, v_1, y) := \frac{1}{|\Omega|}\int_\Omega (\partial_u g_\delta)\big(x, v_1 + v_2(\delta, v_1, y) + \delta w(t, x)\big)\, dx\, dt,$$
$$w = a_2 L^{-1}(v^2) + y. \tag{119}$$

The bounds for the operator D (see Lemma 4.3, Corollary 4.2) still hold, assuming an analogous nonresonance condition, and we can define in the same way the operators $\mathscr{U}, \mathscr{R}_1, \mathscr{R}_2$, with $\|\mathscr{U}^{-1}h\|_{\sigma,s'} = (1 + O(\delta))\|h\|_{\sigma,s'}$. With the same arguments, we obtain for $\mathscr{R}_1$ the bound

$$\|\mathscr{R}_1 h\|_{\sigma,s+(\tau-1)/2} \le \delta^{2-\tau}\frac{C}{\gamma}\|h\|_{\sigma,s+(\tau-1)/2},$$

which is enough since $\tau < 2$.

For the estimate of $\mathscr{R}_2$, the most delicate term to deal with is $\delta^2|D|^{-1/2}D_y F|D|^{-1/2}$, where

$$F(\delta, v_1, y) := \mathscr{R}\big((v_1 + v_2(\delta, v_1, y))^2\big)$$

because the operator $\mathscr{R}$ induces a loss of regularity (see (116)). However, again the regularizing property (117) of the map v_2 enables us to obtain the bound

$$\|\mathscr{R}_2 h\|_{\sigma,s+(\tau-1)/2} \le C\frac{\delta}{\gamma}\ \|h\|_{\sigma,s+(\tau-1)/2}. \tag{120}$$

The key point is that the loss of $(\tau - 1)$ derivatives due to $|D|^{-1/2}$ applied twice, added to the loss of one derivative due to $\mathscr{R}$ in (116), is compensated by the gain of two derivatives with v_2, whenever $\tau < 2$. Let us enter briefly into details:

$$\begin{aligned} \|D_y F(\delta, v_1, y)[h]\|_{\sigma,s+1} &= \big\|2\mathscr{R}\big((v_1+v_2)D_y v_2(\delta, v_1, y)[h]\big)\big\|_{\sigma,s+1} \\ &\le 2\|(v_1+v_2)D_y v_2(\delta, v_1, y)[h]\|_{\sigma,s+2} \\ &\le C\|(v_1+v_2)\|_{\sigma,s+2}\|D_y v_2(\delta, v_1, y)[h]\|_{\sigma,s+2} \\ &\le K(N, R, \|y\|_{\sigma,s})\|h\|_{\sigma,s} \end{aligned}$$

by the regularizing property (117) of v_2. We can then derive (120) as in the proof of Lemma 4.9, using the fact that $\tau < 2$. □

Finally, inserting $\widetilde{y}(\delta, v_1)$ in the $(Q1')$-equation, we get

$$-\Delta v_1 = \mathscr{G}(\delta, v_1), \tag{121}$$

where

$$\mathscr{G}(0, v_1) := -\Pi_{V_1}\big(2a_2(v_1 + v_2(0, v_1, 0))L^{-1}(v_1 + v_2(0, v_1, 0))^2\big).$$

As in Section 5.2, since $\Phi_0 : V \to \mathbf{R}$ possesses a nondegenerate critical point $\overline{v}$, the equation $-\Delta v_1 = \mathscr{G}(0, v_1)$ has the nondegenerate solution $\overline{v}_1 := \Pi_{V_1}\overline{v} \in B(R, V_1)$, and by the implicit function theorem, there exists a smooth path $\delta \mapsto v_1(\delta) \in B(2R, V_1)$ of solutions of (121) with $v_1(0) = \overline{v}$. As in Proposition 3.2, this implies that the set $\mathscr{C} = \{\delta \in (0, \delta_0) \mid (\delta, v_1(\delta)) \in B_\infty\}$ has asymptotically full measure at zero. □

A. Appendix

LEMMA A.1
If q is an even integer, then

$$\int_\Omega a(x)v^q(t,x)\,dt\,dx = 0,\ \forall v \in V \iff \big\{a(\pi - x) = -a(x),\ \forall x \in [0,\pi]\big\}.$$

If $q \ge 3$ is an odd integer, then

$$\int_\Omega a(x)v^q(t,x)\,dt\,dx = 0,\ \forall v \in V \iff \big\{a(\pi - x) = a(x),\ \forall x \in [0,\pi]\big\}.$$

Proof

We first assume that $q = 2s$ is even. If $a(\pi - x) = -a(x)\ \forall x \in (0, \pi)$, then for all $v \in V$,

$$\begin{aligned}\int_\Omega a(x)v^{2s}(t,x)\,dt\,dx &= \int_\Omega a(\pi - x)v^{2s}(t, \pi - x)\,dt\,dx \\ &= \int_\Omega -a(x)\big(-v(t+\pi, x)\big)^{2s}\,dt\,dx \\ &= -\int_\Omega a(x)v^{2s}(t,x)\,dt\,dx,\end{aligned}$$

and so $\int_\Omega a(x)v^{2s}(t,x)\,dt\,dx = 0$.

Now assume that $\Sigma(v) := \int_\Omega a(x)v^{2s}(t,x)\,dt\,dx = 0$, $\forall v \in V$. Writing that $D^{2s}\Sigma = 0$, we get

$$\int_\Omega a(x)v_1(t,x)\cdots v_{2s}(t,x)\,dt\,dx = 0, \quad \forall (v_1, \ldots, v_{2s}) \in V^{2s}.$$

Choosing $v_{2s}(t,x) = v_{2s-1}(t,x) = \cos(lt)\sin(lx)$, we obtain

$$\frac{1}{4}\int_\Omega a(x)v_1(t,x)\cdots v_{2(s-1)}(t,x)\big(\cos(2lt)+1\big)\big(1-\cos(2lx)\big)\,dt\,dx = 0.$$

Taking limits as $l \to \infty$, there results $\int_\Omega a(x)v_1(t,x)\cdots v_{2(s-1)}(t,x)\,dt\,dx = 0$, $\forall (v_1, \ldots, v_{2(s-1)}) \in V^{2(s-1)}$. Iterating this operation, we finally get

$$\forall (v_1, v_2) \in V^2, \quad \int_\Omega a(x)v_1(t,x)v_2(t,x)\,dt\,dx = 0, \qquad \text{and} \qquad \int_0^\pi a(x)\,dx = 0.$$

Choosing $v_1(t,x) = v_2(t,x) = \cos(lt)\sin(lx)$ in the first equality, we derive that $\int_0^\pi a(x)\sin^2(lx)\,dx = 0$. Hence $\forall l \in \mathbf{N}$, $\int_0^\pi a(x)\cos(2lx)\,dx = 0$. This implies that a is orthogonal in $L^2(0,\pi)$ to $F = \{b \in L^2(0,\pi) \mid b(\pi - x) = b(x) \text{ a.e.}\}$. Hence $a(\pi - x) = -a(x)$ a.e., and since a is continuous, the identity holds everywhere.

We next assume that $q = 2s+1$ is odd, $q \geq 3$. The first implication is derived in a similar way. Now assume that $\int_\Omega a(x)v^q(t,x)\,dt\,dx = 0$, $\forall v \in V$. We can prove exactly as in the first part that

$$\forall (v_1, v_2, v_3) \in V^3, \quad \int_\Omega a(x)v_1(t,x)v_2(t,x)v_3(t,x)\,dt\,dx = 0.$$

Choosing $v_1(t,x) = \cos(l_1 t)\sin(l_1 x)$, $v_2(t,x) = \cos(l_2 t)\sin(l_2 x)$, $v_3(t,x) = \cos((l_1 + l_2)t)\sin((l_1+l_2)x)$, and using the fact that $\int_0^{2\pi}\cos(l_1 t)\cos(l_2 t)\cos((l_1+l_2)t)\,dt \neq 0$, we obtain

$$\begin{aligned}\int_0^\pi & a(x)[\sin^2(l_1x)\sin(l_2x)\cos(l_2x)+\sin^2(l_2x)\sin(l_1x)\cos(l_1x)]\,dx \\ &= \int_0^\pi a(x)\sin(l_1x)\sin(l_2x)\sin\big((l_1+l_2)x\big)\,dx = 0.\end{aligned} \tag{122}$$

Letting l_2 go to infinity and taking limits, (122) yields $\int_0^\pi (1/2)a(x)\sin(l_1x)\cos(l_1x)\,dx = 0$. Hence $\int_0^\pi a(x)\sin(2lx) = 0$, $\forall l > 0$. This implies that, in $L^2(0,\pi)$, a is orthogonal to $G = \{b \in L^2(0,\pi) \mid b(\pi - x) = -b(x)$ a.e.$\}$. Hence $a(\pi - x) = a(x)$, $\forall x \in (0,\pi)$. □

Proof of Lemma 4.1
Let $K_k(\varepsilon) = S_k^{-1}(\varepsilon)$ be the self-adjoint compact operator of F_k defined by

$$\langle K_k(\varepsilon)u, v\rangle_\varepsilon = (u,v)_{L^2}, \quad \forall u, v \in F_k.$$

(In other words, $K_k(\varepsilon)u$ is the unique weak solution $z \in F_k$ of $S_k z := u$.)

Note that $K_k(\varepsilon)$ is a positive operator, that is, $\langle K_k(\varepsilon)u, u\rangle_\varepsilon > 0$, $\forall u \neq 0$, and note that $K_k(\varepsilon)$ is also self-adjoint for the L^2-scalar product.

By the spectral theory of compact self-adjoint operators in Hilbert spaces, there is a $\langle\ ,\ \rangle_\varepsilon$-orthonormal basis $(v_{k,j})_{j\geq 1, j\neq k}$ of F_k such that $v_{k,j}$ is an eigenvector of $K_k(\varepsilon)$ associated to a positive eigenvalue $\nu_{k,j}(\varepsilon)$; the sequence $(\nu_{k,j}(\varepsilon))_j$ is nonincreasing and tends to zero as $j \to +\infty$. Each $v_{k,j}(\varepsilon)$ belongs to $D(S_k)$ and is an eigenvector of S_k with associated eigenvalue $\lambda_{k,j}(\varepsilon) = 1/\nu_{k,j}(\varepsilon)$, with $(\lambda_{k,j}(\varepsilon))_{j\geq 1} \to +\infty$ as $j \to +\infty$.

The map $\varepsilon \mapsto K_k(\varepsilon) \in \mathscr{L}(F_k, F_k)$ is differentiable, and $K_k'(\varepsilon) = -K_k(\varepsilon)MK_k(\varepsilon)$, where $Mu := \pi_k(a_0u)$.

For $u = \sum_{j\neq k}\alpha_j v_{k,j}(\varepsilon) \in F_k$,

$$\langle u,u\rangle_\varepsilon = \sum_{j\neq k}|\alpha_j|^2 \qquad \text{and} \qquad (u,u)_{L^2} = \sum_{j\neq k}\frac{|\alpha_j|^2}{\lambda_{k,j}(\varepsilon)}.$$

As a consequence,

$$\begin{aligned}\lambda_{k,j}(\varepsilon) = \min\Big\{ & \max_{u\in F, \|u\|_{L^2}=1}\langle u,u\rangle_\varepsilon;\ F \text{ subspace of } F_k \text{ of dimension } j \text{ (if } j<k), \\ & j-1 \text{ (if } j>k)\Big\}.\end{aligned} \tag{123}$$

It is clear by inspection that $\lambda_{k,j}(0) = j^2$ and that we can choose $v_{k,j}(0) = \sqrt{2/\pi}\sin(jx)/j$. Hence, by (123), $|\lambda_{k,j}(\varepsilon) - j^2| \leq |\varepsilon|\ \|a_0\|_\infty < 1$, from which we derive

$$\forall l \neq j, \quad |\lambda_{k,l}(\varepsilon) - \lambda_{k,j}(\varepsilon)| \geq (l+j) - 2 \geq 2\min(l,j) - 1 \ \ (\geq 1). \tag{124}$$

In particular, the eigenvalues $\lambda_{k,j}(\varepsilon)$ ($\nu_{k,j}(\varepsilon)$) are simple. By the variational characterization (123), we also see that $\lambda_{k,j}(\varepsilon)$ depends continuously on ε, and we can assume, without loss of generality, that $\varepsilon \mapsto v_{k,j}(\varepsilon)$ is a continuous map to F_k.

Let $\varphi_{k,j}(\varepsilon) := \sqrt{\lambda_{k,j}(\varepsilon)}v_{k,j}(\varepsilon)$; $(\varphi_{k,j}(\varepsilon))_{j\neq k}$ is an L^2-orthogonal family in F_k, and

$$\forall\varepsilon, \quad \begin{cases} K_k(\varepsilon)\varphi_{k,j}(\varepsilon) = \nu_{k,j}(\varepsilon)\varphi_{k,j}(\varepsilon), \\ \big(\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\big)_{L^2} = 1. \end{cases}$$

We observe that the L^2-orthogonality with respect to $\varphi_{k,j}(\varepsilon)$ is equivalent to the $\langle\ ,\ \rangle_\varepsilon$-orthogonality with respect to $\varphi_{k,j}(\varepsilon)$, and we observe that $E_{k,j}(\varepsilon) := [\varphi_{k,j}(\varepsilon)]^\perp$ is invariant under $K_k(\varepsilon)$. Using the fact that $L_{k,j} := (K_k(\varepsilon)-\nu_{k,j}(\varepsilon)I)_{|E_{k,j}(\varepsilon)}$ is invertible, it is easy to derive from the implicit function theorem that the maps $(\varepsilon \mapsto \nu_{k,j}(\varepsilon))$ and $(\varepsilon \mapsto \varphi_{k,j}(\varepsilon))$ are differentiable.

Denoting by P the orthogonal projector onto $E_{k,j}(\varepsilon)$, we have

$$\begin{aligned} \varphi'_{k,j}(\varepsilon) &= L^{-1}\big(-PK'_k(\varepsilon)\varphi_{k,j}(\varepsilon)\big) = L^{-1}\big(PK_kMK_k\varphi_{k,j}(\varepsilon)\big) \\ &= \nu_{k,j}(\varepsilon)L^{-1}K_kPM\varphi_{k,j}(\varepsilon), \\ \nu'_{k,j}(\varepsilon) &= \big(K'_k(\varepsilon)\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\big)_{L^2} = -\big(K_kMK_k\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\big)_{L^2} \\ &= -\big(MK_k\varphi_{k,j}(\varepsilon), K_k\varphi_{k,j}(\varepsilon)\big)_{L^2} = -\nu^2_{k,j}(\varepsilon)\big(M\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\big)_{L^2}. \end{aligned} \tag{125}$$

We have

$$\nu_{k,j}L^{-1}K_k\Big(\sum_{l\neq j}\alpha_l v_{k,l}\Big) = \sum_{l\neq j}\frac{\nu_{k,j}\nu_{k,l}}{\nu_{k,l}-\nu_{k,j}}\alpha_l v_{k,l} = \sum_{l\neq j}\frac{\alpha_l}{\lambda_{k,j}-\lambda_{k,l}}v_{k,l}.$$

Hence, by (124), $|\nu_{k,j}L^{-1}K_kPu|_{L^2} \le |u|_{L^2}/j$. We obtain $|\varphi'_{k,j}(\varepsilon)|_{L_2} = O(|a_0|_\infty/j)$. Hence

$$\Big|\varphi_{k,j}(\varepsilon) - \sqrt{\frac{2}{\pi}}\sin(jx)\Big|_{L^2} = O\Big(\frac{\varepsilon|a_0|_\infty}{j}\Big).$$

Hence, by (125),

$$\begin{aligned} \lambda'_{k,j}(\varepsilon) &= \big(M\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\big)_{L^2} = \int_0^\pi a_0(x)(\varphi_{k,j})^2\,dx \\ &= \frac{2}{\pi}\int_0^\pi a_0(x)\big(\sin(jx)\big)^2\,dx + O\Big(\frac{\varepsilon|a_0|^2_\infty}{j}\Big). \end{aligned}$$

Writing $\sin^2(jx) = (1-\cos(2jx))/2$, and since $\int_0^\pi a_0(x)\cos(2jx)\,dx = -\int_0^\pi (a_0)_x(x)\sin(2jx)/2j\,dx$, we get

$$\lambda'_{k,j}(\varepsilon) = \frac{1}{\pi}\int_0^\pi a_0(x)\,dx + O\Big(\frac{\|a_0\|_{H^1}}{j}\Big) = M(\delta, v_1, w) + O\Big(\frac{\|a_0\|_{H^1}}{j}\Big).$$

Hence $\lambda_{k,j}(\varepsilon) = j^2 + \varepsilon M(\delta, v_1, w) + O(\varepsilon \|a_0\|_{H^1}/j)$, which is the first estimate in (80). □

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